3.7 ADHM Construction for Most General Multi-Instanton Solutions

□ **Hitorical Remarks**:

The introduction section of the paper by Corrigan and Goddard (Ann. Phys. 154 (84) 253) summarizes the history of the ADHM construction.

1. ADHM construction is based on the observation of Ward(R.S. Ward, PLA61 (77) 81; M.F. Atiyah and R.S. Ward, CMP55 (77) 117) that Penrose's twistor theory can be used to establish a correspondence between self-dual gauge fields and certain analytic vector bundles.

    - Original proof of construction by ADHM: Given a self-dual gauge field, the analytic vector bundle was constructed using a sequence of vector spaces defined in terms of sheaf cohomology groups.
    - Twistor theory: Such groups $\leftrightarrow$ spaces of solutions of covariant diff. eq. such as Dirac and Laplace equations.
    - Exploiting this idea, Witten outlined how the sequence of vector spaces used by ADHM may be defined i.t.o. spaces of solutions to diff. eq.
    - This approach was elaborated by Osborn (CMP86 (82) 195) partly to build new self-dual monopole solutions.

2. Building on this and on his own description of the BPS monopole (PLB90(80) 413), Nahm obtained the adaptation of the construction to monopole solutions.

3. ADHM construction was rephrased in elementary language in

   - [CFGT] Corrigan, Fairlie, Goddard and Templeton, NPB140 (78) 31
   - [CSW] Chirst, Stanton and Weinberg, PRD18 (78) 2013

   However, these papers only described **how to construct solutions from a set of matrices. No prescription for the converse problem.**

4. Nahm gave expressions for these matrices i.t.o. solutions of Dirac and Laplace equations in the background of instantons.

5. Corrigan and Goddard (Ann. Phys. 154 (84) 253) gave an elementary proof of the completeness of ADHM construction.

□ **Remarks on the formalism for different gauge groups**:

Although the general idea is the same, **explicit descriptions of ADHM construction for different gauge groups, namely $\boldsymbol{Sp(n)}$, $\boldsymbol{SU(n)}$ and $\boldsymbol{O(n)}$, differ in details.**

- The most detailed analysis, starting from $O(n)$ group and extending to other cases, is given in [CSW].

- On the other hand, the connection to the original idea of ADHM is sketched in [CFGT].

♡ **Use of quaternions becomes extremely convenient.**

3.7.1 Construction for $Sp(1) \simeq SU(2)$

We begin with the simplest case of $Sp(1) \simeq SU(2)$, which can be generalized for $Sp(n)$ without much modifications.

□ **A brief review of $Sp(n)$:**

The compact real form of $Sp(n)$ can be thought of as the group of $\boldsymbol{n \times n}$ **quaternionic unitary** matrices U.
Especially, for $n = 1$, U is simply a single quaternion with the unitarity condition $U^\dagger U = 1$. But this is nothing but the definition of $SU(2)$ matrices. Thus $Sp(1) \simeq SU(2)$.

Let U be an $n \times n$ quaternionic matrix. U and its conjugate can be written as

$$\begin{aligned} U_{ij} &= U_{ij}^\mu e_\mu \\ (U^\dagger)_{ij} &\equiv U_{ji}^\mu e_\mu^\dagger \\ U_{ij}^\mu &= \text{real} \end{aligned}$$

Let us write $V_{ij} \equiv (U^\dagger U)_{ij}$, which is a quaternion. Then,

$$\begin{aligned} (V_{ij})^\dagger &= (U_{ki}^\mu U_{kj}^\nu e_\mu^\dagger e_\nu)^\dagger \\ &= U_{ki}^\mu U_{kj}^\nu e_\nu^\dagger e_\mu = (U^\dagger U)_{ji} = V_{ji} \end{aligned}$$

Writing $V_{ij} = V_{ij}^\mu e_\mu$, this means $V_{ij}^\mu e_\mu^\dagger = V_{ji}^\mu e_\nu$ and hence

$$\boxed{V_{ij}^0 = V_{ji}^0\,, \quad V_{ij}^a = -V_{ji}^a}$$

Therefore, V^0 is symmetric while V^a are antisymmetric.
Thus, it is useful to remember that for any U_{ij},

$$\boldsymbol{(U^\dagger U)_{ij} = \textbf{symmetric} \quad \Leftrightarrow \quad \textbf{real}}$$

Now the unitarity condition reads

$$V^0_{ij} \;=\; \delta_{ij}\,, \quad V^a_{ij} = 0$$

They give, respectively, $\frac{1}{2}n(n+1)$ and $\frac{3}{2}n(n-1)$ conditions and hence altogether $2n^2 - n$ real conditions. Since the original U^μ_{ij} has $4n^2$ real DOF, the real dimension of $Sp(n)$ is

$$\mathbf{dim}_{\boldsymbol{R}}\, \boldsymbol{Sp(n)} \;=\; \boldsymbol{4n^2 - (2n^2 - n) = 2n^2 + n}$$

□ **Construction of the instanton solution for $\boldsymbol{Sp(1)}$:**

To construct multi-instanton solution with instanton number k, introduce the following **matrix $\boldsymbol{\Delta(x)}$**:

- **$\boldsymbol{\Delta(x)}$ is a $\boldsymbol{(k+1)\times k}$ matrix made out of quaternions linear in $\boldsymbol{x \equiv x_\mu e_\mu}$.** More explicitly,

$$\boxed{\begin{aligned} \boldsymbol{\Delta_{\lambda i}(x)} &= \boldsymbol{a_{\lambda i} + b_{\lambda i} x \equiv \Delta^\mu_{\lambda i}(x) e_\mu} \\ \boldsymbol{a_{\lambda i}} &= \boldsymbol{a^\mu_{\lambda i} e_\mu}\,, \qquad \boldsymbol{b_{\lambda i} = b^\mu_{\lambda i} e_\mu} \\ \boldsymbol{\lambda} &= \boldsymbol{0 \sim k}\,, \quad \boldsymbol{i = 1 \sim k} \end{aligned}}$$

  $a^\mu_{\lambda i}, b^\mu_{\lambda i}$ are all **real**, since $a_{\lambda i}, b_{\lambda i}$ are quaternions.
  Matrix $\boldsymbol{a}$ must be of **rank $\boldsymbol{k}$** in order for f^{-1} defined below is invertible at $\underline{x = 0}$.
  Matrix $\boldsymbol{b}$ should also be of **rank $\boldsymbol{k}$** to describe a configuration with instanton number k and not less.

- Define $\Delta^\dagger$ as follows:

$$(\boldsymbol{\Delta}^\dagger)_{i\lambda} \equiv \boldsymbol{\Delta}^\mu_{\lambda i} \boldsymbol{e}^\dagger_\mu \tag{3.1}$$

  In other words, we transpose the (λi) indices and take the quaternionic conjugate for e_μ. This is the same rule as the definition of $(U^\dagger)_{ij}$.

- **Quadratic constraints:** $\Delta(x)$ is assumed to satisfy the following quadratic constraints:

$$\boxed{\boldsymbol{\Delta}^\dagger(\boldsymbol{x})\boldsymbol{\Delta}(\boldsymbol{x}) = \boldsymbol{f}^{-1} = \textbf{real, non-singular } \boldsymbol{k} \times \boldsymbol{k} \textbf{ matrix}}$$

  (As we remarked earlier, we may instead demand that $(\Delta^\dagger(x)\Delta(x))_{ij}$ be symmetric and non-singular.) Since the elements of f^{-1} are real (*i.e.* $\propto e_0$), **it commutes with quaternions**. This will be important later.

Now to construct the anti-self-dual gauge field, find a $(k+1)$-dimensional quaternionic vector $v(x)$ satisfying the following two conditions:

$$\boxed{\begin{array}{ll} \boldsymbol{(i)} & \boldsymbol{v}^\dagger(\boldsymbol{x})\boldsymbol{\Delta}(\boldsymbol{x}) = \boldsymbol{0} \\ \boldsymbol{(ii)} & \boldsymbol{v}^\dagger(\boldsymbol{x})\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{1} \end{array}}$$

The above conditions
(i) imposes k quaternionic conditions and the solution can always be found.
(ii) then gives a normalization condition.
Clearly, $v(x)$ is determined up to an $Sp(1)$ gauge transformation

$$\boxed{v(x) \to v(x)g(x)\,, \quad g(x) \in Sp(1)\,,\ g^\dagger g = 1}$$

It is obvious that (i) and (ii) are invariant under this transformation.

Now define the gauge field $A_\mu(x)$ by

$$\boxed{\boldsymbol{A_\mu(x) \;=\; v^\dagger(x)\partial_\mu v(x)}}$$

A_μ is not a pure gauge since $v(x)$ is $(k+1)\times 1$.

Anti-hermiticity: Anti-hermiticity of A_μ is easily checked using the normalization condition above:

$$A_\mu^\dagger \;=\; (v^\dagger\partial_\mu v)^\dagger = \partial_\mu v^\dagger v = -v^\dagger\partial_\mu v = -A_\mu$$

Tracelessness:

$$\begin{aligned} \mathrm{Tr}A_\mu &= \mathrm{Tr}(A_{\mu,\sigma}e_\sigma) = A_{\mu,0} \\ &= -\mathrm{Tr}A_\mu^\dagger = -A_{\mu,0} \\ \therefore \qquad \mathrm{Tr}A_\mu &= 0 \end{aligned}$$

Thus, A_μ properly belongs to $SU(2)$. It is easy to see that $v(x) \to v(x)g(x)$ simply induces a gauge transformation of $A_\mu(x)$.

Proof of self-duality of $F_{\mu\nu}$: Substituting A_μ above into the definition of $F_{\mu\nu}$, we get

$$\begin{aligned} F_{\mu\nu} &= \partial A_\mu + A_\mu A_\nu - (\mu \leftrightarrow \nu) \\ &= \partial_\mu(v^\dagger \partial_\nu v) + v^\dagger \partial_\mu v v^\dagger \partial_\nu v - (\mu \leftrightarrow \nu) \\ &= \partial_\mu v^\dagger \partial_\nu v + v^\dagger \partial_\mu v v^\dagger \partial_\nu v - (\mu \leftrightarrow \nu) \end{aligned} \tag{3.2}$$

Using $v^\dagger \partial_\mu v = -\partial_\mu v^\dagger v$, this can be written as

$$F_{\mu\nu} = \partial_\mu v^\dagger P \partial_\nu v - (\mu \leftrightarrow \nu) \tag{3.3}$$

$$\text{where} \qquad P \equiv 1 - v v^\dagger \tag{3.4}$$

Due to the "normalization condition" $v^\dagger v = 1$, it is easy to verify that $(k+1) \times (k+1)$ matrix P is a **projection operator**, *i.e.* $P^2 = P$. Clearly, **it projects to the space orthogonal to v** since

$$Pv = (1 - vv^\dagger)v = v - v = 0.$$

What is important is that this projection operator can be written in terms of Δ as

$$\boxed{\boldsymbol{P = \Delta(x) f \Delta^\dagger(x)} \qquad (3.5)}$$

This is checked as follows:

- It is obviously a projector:

$$\Delta f \Delta^\dagger \Delta f \Delta^\dagger = \Delta f f^{-1} f \Delta^\dagger = \Delta f \Delta^\dagger \tag{3.6}$$

- **It projects to the space orthogonal to $\boldsymbol{v}$**, due to the condition $\Delta^\dagger v = 0$:

$$\Delta f \Delta^\dagger v = 0 \tag{3.7}$$

Now we want to let the derivative ∂_μ act on $\Delta(x)$ rather than on v's since $\partial_\mu \Delta(x)$ is simple. By differentiating the condition $\Delta^\dagger v = 0$, we get

$$\partial_\mu \Delta^\dagger v = -\Delta^\dagger \partial_\mu v \tag{3.8}$$

$$v^\dagger \partial_\mu \Delta = -\partial_\mu v^\dagger \Delta \tag{3.9}$$

Using these relations, $F_{\mu\nu}$ can be rewritten as

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu v^\dagger \Delta f \Delta^\dagger \partial_\nu v - (\mu \leftrightarrow \nu) \\ &= v^\dagger \partial_\mu \Delta f \partial_\nu \Delta^\dagger v - (\mu \leftrightarrow \nu) \end{aligned}$$

Now we use the fact that $\Delta(x)$ is linear in x:

$$\partial_\mu \Delta = b e_\mu \,, \qquad \partial_\nu \Delta^\dagger = e_\nu^\dagger b^\dagger \tag{3.10}$$

Thus using the fact that **$\boldsymbol{f}$ commutes with quaternions**, we get

$$\begin{aligned} F_{\mu\nu} &= (v^\dagger b e_\mu f e_\nu^\dagger b^\dagger v) - (\mu \leftrightarrow \nu) \\ &= v^\dagger b f (e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger) b^\dagger v \\ &= 2i\eta_{a\mu\nu} v^\dagger b f \tau_a b^\dagger v \end{aligned}$$

which is self-dual.

Clearly, to get an anti-self-dual solution, all we have to do is to

take instead $\Delta = a + bx^\dagger$.
We will count the number of degrees of freedom(DOF) up to a gauge transformation (*i.e.* the dimension of the moduli space) for this solution after generalizing this construction to $Sp(n)$ case.

3.7.2 Construction for $Sp(n)$

The above construction for $Sp(1)$ case can be straightforwardly generalized for $Sp(n)$. In this case, take

$$\begin{aligned} \Delta_{\lambda i}(x) &= a_{\lambda i} + b_{\lambda i} x \\ -(n-1) &\leq \lambda \leq k\,, \quad 1 \leq i \leq k \end{aligned}$$

In other words, $\Delta(x)$ is now a **$(k+n)\times k$ quaternionic** matrix. The quadratic condition is also of the same type:

$$\Delta^\dagger(x)\Delta(x) = f^{-1} = k \times k \text{ symmetric, non-singular}$$

This is the same as demanding

$$a^\dagger a\,, \quad b^\dagger b\,, \quad a^\dagger b\,, \quad b^\dagger a \quad \text{all } k \times k \text{ symmetric}$$

Introduce $(n+k) \times n$ quaternionic matrix $v(x)$ and demand

$$\begin{aligned} v^\dagger(x)\Delta(x) &= 0 \\ v^\dagger(x)v(x) &= \mathbf{1}_n \end{aligned}$$

This time, such a $v(x)$ is determined up to $Sp(n)$ (unitary quaternionic) gauge transformation $v(x) \to v(x)\gamma(x)$, which leaves all the conditions invariant. The rest of the procedure is exactly the same as for the $Sp(1)$ case.

□ **Canonical form of a and b and counting of the parameters**:

To count the dimension of the moduli space, we first bring the matrices a and b into a simple **canonical form**, using transformations which **preserve all the constraints and the definition $\boldsymbol{A_\mu = v^\dagger \partial_\mu v}$**. The allowed transformations are of the type

$$
\begin{aligned}
a &\rightarrow QaK\,, \qquad b \rightarrow QbK \\
v &\rightarrow Qv
\end{aligned}
$$

where

$$
\begin{aligned}
Q &= (n+k)\times(n+k) \text{ unitary quaternionic} \\
K &= k \times k \text{ real non-singular}
\end{aligned}
$$

Unitarity of $\boldsymbol{Q}$ is required to preserve $v^\dagger v = 1$.
Real non-singular nature of $\boldsymbol{K}$ is needed to preserve the real symmetric non-singular nature of $a^\dagger a$, $b^\dagger b$ etc.

Now we proceed in steps:

1. $b^\dagger b$ is real symmetric and transformed to $K^T b^\dagger b K$ under the above transformations. Thus, we may use K to bring $b^\dagger b$ to $\mathbf{1}_k$. (Diagonalize and then rescale the diagonal entries to 1. Note that the scale of K is not restricted.)

2. Next Choose Q to bring b itself to the following form:

$$
\begin{aligned}
b_{\lambda i} &= \delta_{\lambda i} \\
b &= \begin{pmatrix} 0_n \\ \mathbf{1}_k \end{pmatrix}
\end{aligned}
$$

   In other words, we choose the part of Q such that $Q^\dagger \begin{pmatrix} 0_n \\ \mathbf{1}_k \end{pmatrix} = b$.

3. Now one can easily check that this form of b is preserved by the following transformations

$$Q = \begin{pmatrix} Q_n & 0 \\ 0 & X \end{pmatrix}, \quad K = X^T$$

$$\text{where} \qquad Q_n \in Sp(n), \quad X \in O(k)$$

Indeed

$$\begin{pmatrix} Q_n & 0 \\ 0 & X \end{pmatrix} \begin{pmatrix} 0_n \\ \mathbf{1}_k \end{pmatrix} X^T = \begin{pmatrix} 0_n \\ X \end{pmatrix} X^T = \begin{pmatrix} 0_n \\ \mathbf{1}_k \end{pmatrix} \qquad (3.11)$$

Then, $a^\dagger a$ gets transformed to $Xa^\dagger aX^T$. So by choosing X approriately, **we can diagonalize $a^\dagger a$**.

4. Write a in the form

$$a = \begin{pmatrix} \rho \\ -y \end{pmatrix}, \quad \rho = n \times k, \quad y = k \times k$$

Then, $b^\dagger a = -y$. This is demanded to be symmetric. Hence $y^T = y$.

5. Thus, the constraints are reduced to

$$(i) \quad a^\dagger_{\lambda i} a_{\lambda j} = 0 \quad \text{for } i \neq j$$
$$(ii) \quad y_{ij} = y_{ji}$$

The first condition says that off-diagonal elements of $a^\dagger a$ is zero.

Now the counting of DOF is easy. First, originally we have

$$\dim_R a = 4(n+k)k = 4nk + 4k^2$$

The number of quaternionic constraints from (i) and (ii) are

$$(i) \quad \frac{1}{2}k(k-1)$$
$$(ii) \quad \frac{1}{2}k(k-1)$$

Together the number of real constrantis are $4k(k-1)$. Therefore the real DOF remaining is

$$4nk + 4k^2 - 4k^2 + 4k = 4nk + 4k.$$

6. Finally, we must take into account the fact that **Q_n part of the transformation still preserves the constrants**. Thus, we must subtract $\dim_R Sp(n) = 2n^2 + n$. So we get

$$\boxed{\#\ \textbf{of moduli} \quad = \quad \boldsymbol{4(n+1)k - (2n^2+n)}} \qquad (3.12)$$

Especially, for $n = 1$ this gives $\boldsymbol{8k-3}$.

□ **Explicit solution for $k = 1$ for $Sp(1)$:**

Start from the canonical form (y will be the position of the instanton)

$$\begin{aligned} a &= \begin{pmatrix} \rho \\ -y \end{pmatrix}, \quad b = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \therefore \qquad \Delta(x) &= a + bx = \begin{pmatrix} \rho \\ x - y \end{pmatrix} \end{aligned}$$

where ρ and y are quaternions. The remaining constraints are

$$\begin{aligned} v(x) &= \begin{pmatrix} v_1(x) \\ v_2(x) \end{pmatrix} \\ (i) \quad \Delta^\dagger(x)v(x) &= \rho^\dagger v_1 + (x^\dagger - y^\dagger)v_2 = 0 \end{aligned}$$

Therefore we can solve for v_2:

$$v_2 = -q(x)v_1 \,, \quad q(x) \equiv (x^\dagger - y^\dagger)^{-1}\rho^\dagger \tag{3.13}$$

The other constraint is the normalization condition

$$(ii) \quad v_1^\dagger v_1 + v_2^\dagger v_2 = 1$$

Let us write $v_1(x) = \frac{1}{\sqrt{\xi(x)}}\chi(x)$, where ξ is real and χ is a unit quaternion. By a gauge transformation, we may set $\chi = 1$ and v becomes

$$v \;=\; \frac{1}{\sqrt{\xi}}\begin{pmatrix} 1 \\ -q \end{pmatrix}$$

Then, the normalization condition reads

$$\begin{aligned} 1 &= \frac{1}{\xi}(1 + q^\dagger q) \\ \therefore \qquad \xi &= 1 + q^\dagger q = \frac{\rho^2 + |x-y|^2}{|x-y|^2} \end{aligned}$$

where $|x-y|^2 = (x-y)^\dagger(x-y) = (x-y)_\mu(x-y)_\mu$.

Let us compute explicitly the projection operator P. From the expression $P = 1 - vv^\dagger$, we get after a simple calclulation,

$$P \;=\; \frac{1}{\xi}\begin{pmatrix} q^\dagger q & q^\dagger \\ q & 1 \end{pmatrix}$$

On the other hand,

$$\Delta^\dagger\Delta \;=\; |\rho|^2 + |x-y|^2 = f^{-1} \tag{3.14}$$

Therefore $\Delta f \Delta^\dagger$ gives

$$\Delta f \Delta^\dagger = f|x-y|^2\begin{pmatrix} q^\dagger q & q^\dagger \\ q & 1 \end{pmatrix}$$

Thus, indeed this conicides with P.

According to the general formula, the field strength is given by

$$F_{\mu\nu} = 2i\eta_{a\mu\nu}(v^\dagger b)f\tau_a(b^\dagger v) = 2i\eta_{a\mu\nu}\frac{f}{\xi}q^\dagger\tau_a q$$

To bring it to a more familiar form, we can make a gauge transformation to remove $q^\dagger$ and q, except for its norm given by $q^\dagger q = \rho^2/|x-y|^2$. Thus after this gauge transformation, $F_{\mu\nu}$ is easily seen to become

$$\boldsymbol{F_{\mu\nu} = \frac{2i\eta_{a\mu\nu}\tau_a\rho^2}{(\rho^2+|x-y|^2)^2}}$$

This is precisely the one-instanton solution obtained previously.

Note that as was already explained in the general procedure, by using the "Q_1" transformation, ρ itself can be made real. Thus it actually describes one DOF, which is the size of the instanton.

Exercise: Construct $k=2$ solution by similar method.

3.7.3 Construction for $SU(n)$

The basic idea is entirely similar and one can guess what would be the correct ansatz:

- To make the gauge group to be $SU(n)$, we want $v(x)$ to be a $m \times n$ **complex** matrix, where m is not yet specified.
- To find the value of m, consider the following. **When $n = 2$, we should recover the previous case for $Sp(1)$.** Since a quaternion can be represented by a 2×2 matrix, $(k+1) \times 1$ quaternionic $v(x)$ for $Sp(1)$ should be regarded as $2(k+1) \times 2 = (2k+2) \times 2$ complex matrix for $SU(2)$

case. This suggests that **$v(x)$ for $SU(n)$ should be a $(2k+n)\times n$ complex matrix.** This is reasonable since $SU(n)$ gauge transformation should act on $v(x)$ from the right.

- This in turn demands that **$\Delta(x)$** to be of $(2k+n)\times l$ type. For $Sp(1)$ it was $(k+1)\times k$ quaternionic containing $4(k+1)k$ real components. The same number of components is contained in $(2k+2)\times k$ complex matrix. Hence **for general n it should be $(2k+n)\times k$ complex.**

- Recall that the self-duality neatly came out from the structure $e_\mu e_\nu^\dagger - e_\nu e_\mu^\dagger$. Thus, we must still make use of quaternionic structure somewhere. The fact that two sets of complex numbers is needed to construct a quaternion suggests that we should consider **2 sets of $\Delta(x)$**, namely, $\Delta(x)_A, A = 1, 2$.

This leads us to the following **ansatz for $SU(n)$**:

Consider 2 sets of $(2k+n)\times k$ **complex** matrices a_A, b_A, $(A = 1, 2)$ and further define $\Delta(x)_A$ linear in $x = x_\mu e_\mu$ as

$$\boldsymbol{\Delta(x)_A = a_A + b_B x_{BA} = a_A + b_B (e_\mu)_{BA} x_\mu} \tag{3.15}$$

$$\boldsymbol{\Delta(x)^\dagger_A = a^\dagger_A + b^\dagger_B (e_\mu)^*_{BA} x_\mu = a^\dagger_A + x_\mu (e^\dagger_\mu)_{AB} b^\dagger_B} \tag{3.16}$$

One may regard the A index as an additional column index so that a, b, Δ may also be considered as $(2k+n)\times 2k$ matrices.

$\Delta(x)$ is assumed to satisfy the following constraints:

- $\Delta(x)^\dagger\Delta(x)$ regarded as a $2k\times 2k$ matrix is invertible for all

x. Thus we may define its inverse as

$$f(x) \equiv \left(\Delta(x)^\dagger \Delta(x)\right)^{-1} \tag{3.17}$$

- It commutes with the quaternions in the sense that

  $$\Delta^\dagger_A \Delta_B q_{BC} = q_{AB} \Delta^\dagger_B \Delta_C \tag{3.18}$$

  This means that $\Delta^\dagger_A \Delta_B \propto \delta_{AB}$ for any x.

Let us now spell out what this condition means for the matrices a and b. We have

$$\Delta^\dagger_A \Delta_B = a^\dagger_A a_B + a^\dagger_A (bx)_B + (x^\dagger b)_A a_B + (x^\dagger b^\dagger)_A (bx)_B$$

We demand this to take the form $C(x)\delta_{AB}$.

- From the x independent part, we get $a^\dagger_A a_B = \mu \delta_{AB}$. By contracting with δ_{AB} we immediately find that $\mu = (1/4) a^\dagger_A a_A$ and hence it must be **hermitian**.

- For the part quadratic in x, we may write $(x^\dagger b^\dagger)_A (bx)_B = C\delta_{AB}$. Contracting with δ_{AB}, we get $4C = x^2 (b^\dagger_A b_A)$, which again must be hermition. Therefore, the condition is $b^\dagger_A b_B = \nu \delta_{AB}$ where ν is **hermitian**. It is clear that this condition in tern gurantees the form $(x^\dagger b^\dagger)_A (bx)_B = C\delta_{AB}$.

- Next demand the term linear in x to be diagonal. This can be written as

  $$a^\dagger_C (bx)_B + (bx^*)_C a_B = c\delta_{CB}$$

  Now use the fundamental identity $\boldsymbol{x^* = \epsilon x \epsilon^T}$ **and contract with** $\boldsymbol{\epsilon_{AC}}$. We get

  $$(\epsilon a^\dagger)_A b_D x_{DB} + (\epsilon b^\dagger)_D x_{DA} a_B = c\epsilon_{AB}$$

Now differentiate with respect to x_{DE}. Then,

$$(*) \quad (\epsilon a^\dagger)_A b_D \delta_{BE} + (\epsilon b^\dagger)_D \delta_{AE} a_B = c_{DE} \epsilon_{AB}$$

Now set $A = B$ and sum. We obtain

$$(**) \quad (\epsilon a^\dagger)_E b_D + (\epsilon b^\dagger)_D a_E = 0$$

Contrarily, once this relation holds, then LHS of $(*)$ becomes antisymmetric in (A, B) and must be of the form of the RHS. In otherwords, $(**)$ is precisely the condition we seek.

Summarizing, **the quadratic conditions for $\boldsymbol{a_A}$ and $\boldsymbol{b_A}$** are

$$\boldsymbol{(i) \quad a_A^\dagger a_B = \mu \delta_{AB} \,, \quad \mu = \textbf{hermitian}} \tag{3.19}$$
$$\boldsymbol{(ii) \quad b_A^\dagger b_B = \nu \delta_{AB} \,, \quad \nu = \textbf{hermitian}} \tag{3.20}$$
$$\boldsymbol{(iii) \quad (\epsilon a^\dagger)_A b_B + (\epsilon b^\dagger)_A a_B = 0} \tag{3.21}$$

□ **Construction of the self-dual solution**:

Given these matrices, one can construct a self-dual solution in the standard way. Define a $(2k+n) \times n$ complex matrix $v(x)$ which satisfies

$$\Delta^\dagger v = 0 \tag{3.22}$$
$$v^\dagger v = \mathbf{1}_n \tag{3.23}$$

$n \times n$ anti-hermitian Yang-Mills potential A_μ is then given by

$$A_\mu(x) \;=\; v(x)^\dagger \partial_\mu v(x) \tag{3.24}$$

Anti-hermiticity is easily checked by using $\partial_\mu v^\dagger v = -v^\dagger \partial_\mu v$, which follows from $v^\dagger v = 1$ above.
A_μ simply undergoes a gauge transformation under the right action

$$v(x) \;\longrightarrow\; v(x)g(x) \qquad g(x) \in U(n) \tag{3.25}$$

In general, A_μ is not yet traceless. However, since A_μ is anti-hermitian, $\mathrm{Tr}A_\mu$ is pure imaginary. This can be removed by the $U(1)$ part of the gauge transformation $v \to ve^{-i\theta\mathbf{1}_n}$

$$A'_\mu \;=\; A_\mu + e^{i\theta}\partial_\mu e^{-i\theta} = A_\mu - i\theta$$

Thus, the remaining gauge transformation is indeed $SU(n)$.

Demonstration of the self-duality of $F_{\mu\nu}$ goes through as before using the projection opertor $P = 1 - vv^\dagger = \Delta f \Delta^\dagger$ and one arrives at

$$F_{\mu\nu} \;=\; v^\dagger \partial_\mu \Delta_A f \partial_\nu \Delta_A^\dagger v - (\mu \leftrightarrow \nu) \tag{3.26}$$

Now we use the fact that $\Delta(x)$ is linear in x:

$$\partial_\mu \Delta_A \;=\; b_\beta (e_\mu)_{BA}\,, \qquad \partial_\nu \Delta_A^\dagger = (e_\nu^\dagger)_{AC} b_C^\dagger \tag{3.27}$$

Thus using the fact that f is diagonal in the spinor indices, we get

$$\begin{aligned} F_{\mu\nu} &= (v^\dagger b_A f b_B^\dagger v)(e_\mu e_\nu^\dagger)_{AB} - (\mu \leftrightarrow \nu) \\ &= 4i(v^\dagger b_A f b_B^\dagger v)(\sigma_{\mu\nu})_{AB} \end{aligned} \tag{3.28}$$

which is self-dual.

Clearly, to get an anti-self-dual solution, all we have to do is to take instead $\Delta_A = a_A + b_B (x^\dagger)_{BA}$.

□ **Canonical form and counting of parameters**:

The transformations which preserve various constraints are of the form

$$\begin{aligned} a &\to QaK\,, \quad b \to QbK\,, \quad v \to Qv \\ \text{where} \qquad Q &= (n+2k)\times(n+2k) \text{ unitary} \qquad (3.29) \\ K &= 2k \times 2k \text{ complex non-singular} \qquad (3.30) \end{aligned}$$

Just as for the case of $Sp(n)$, we perform the following simplifications:

- Use K to bring $b^\dagger b$ into $\mathbf{1}_{2k}$.
- Use Q to bring b itself to the form

$$b = \begin{pmatrix} 0_{n,k} & 0_{n,k} \\ \mathbf{1}_k & 0 \\ 0 & \mathbf{1}_k \end{pmatrix} \qquad (3.31)$$

  The transformations which preserves this form of b is

$$\begin{aligned} QbK &= \begin{pmatrix} Q_n & 0 \\ 0 & K^{-1} \end{pmatrix} \begin{pmatrix} 0_{n,2k} \\ \mathbf{1}_{2k} \end{pmatrix} K = \begin{pmatrix} 0_{n,2k} \\ \mathbf{1}_{2k} \end{pmatrix} \\ Q_n &\in U(n)\,, \qquad K = 2k \times 2k \text{ complex} \qquad (3.32) \end{aligned}$$

- Now write a in the form

$$a = \begin{pmatrix} \rho_1 & \rho_2 \\ -y_{11} & -y_{12} \\ -y_{21} & -y_{12} \end{pmatrix} \qquad (3.33)$$

  where ρ_i are $n \times k$, while y_{AB} are $k \times k$.

- Impose the quadratic condition involving $b^\dagger a$ and $a^\dagger b$. After a simple calculation, this leads to

$$(i) \quad y_{22} \;=\; y_{11}^\dagger \,, \qquad y_{21} = -y_{12}^\dagger$$

Note this is precisely the condition that **the lower $2k \times 2k$ block of a can be regarded as a $k \times k$ quaternionic matrix $-y$**.

- K transforms $a^\dagger a$ to $K^\dagger a^\dagger a K$. Thus by choosing K to be an appripriate unitary transformation, we may diagonalize the hermitian matrix $a^\dagger a$

$$(ii) \quad a^\dagger_{\lambda,iA} a_{\lambda,jB} = \mu_i \delta_{ij} \delta_{AB} \,, \quad \mu_i = \text{real}$$

- Now we look for the remaining transformation of the type (3.32) which preserves both the canonical form and the constraints.

Counting of DOF:

- Originally a has $(2k+n) \times 2k \times 2 = 4nk + 8k^2$ real DOF.
- (i) imposes $k^2 \times 2 \times 2 = 4k^2$ real constraints
- Since $a^\dagger a$ is hermitian, (ii) imposes $\frac{1}{2} 2k(2k-1) \times 2$ off diagonal and k real constraints (the values of μ_i). This adds up to $4k^2 - k$ constraints.
- Thus, so far, we have $4nk + k$ DOF.
- We must still subtract the dimension of the invariance group. Q_n part still gives $U(n)$. K which preserves (ii) is $U(1)^k$ but the overall $U(1)$ is trivial and we have $U(1)^{k-1}$.

Combining altogether we get the following result:

(1) For $2k \geq n$, the $U(n)$ group acts faithfully and the genuine DOF is

$$4nk + k - (n^2 + k - 1) = 4nk - n^2 + 1$$

For n=2, we get $8k - 3$, which is the correct number.
(2) On the other hand, for $2k < n$, Q_{n-2k} part of Q_n does not reduce the number of parameters (redundant). Thus we must add back its dimension *i.e.* $(n-2k)^2 = n^2+4k^2-4nk$, obtaining

$$4nk - n^2 + 1 + n^2 + 4k^2 - 4nk = 4k^2 + 1$$

This is also known to be the correct result.

□ **Explicit Construction for $SU(2), k = 1$:**

Now let us see how the ***SU*(2) ansatz** leads to the previous $k = 1$ solution.

Canonical form of a and b can be written as

$$b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \qquad a = \begin{pmatrix} \rho \\ -y \end{pmatrix}$$

where ρ is a 2×2 complex matrix and y is a quaternion. This is quite similar to the $Sp(1)$ ansatz, except ρ is not a quaternion. However, due to the quadratic constraints $a^\dagger a = \rho^\dagger \rho + y^2 =$ diagonal both in the i and A indices, $\rho^\dagger \rho$ must be of the form $\rho^2 \mathbf{1}_2$, where ρ now is a real number. Thus as far as a, b and hence Δ is concerned, we have effectively the same expression as for the $Sp(1)$ ansatz.

As for v, it is originally of the form

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

where v_i is 2×2 compex. We may however, use $U(2)$ transformation Q_2 to act from left on v_1 and removes 4 of the 8 real DOF. Thus, effectively, v_1 can be regarded as a quaternion again. Then, the condition $v^\dagger \Delta = 0$ gives the relation

$$v_2 = -(x^\dagger - y^\dagger)^{-1} \rho^\dagger v_1$$

which is the same as the previous case. Therefore, by this relation, v_2 can also be regarded as a quaternion. The rest of the procedure is the same.

The only apparent difference is in the last step, where we encounter $b_A (\sigma_{\mu\nu})_{AB} b_B^\dagger$. However, since b_A are unit basis vectors in 2-dimensions, this represents nothing but the matrix $\sigma_{\mu\nu}$ itself. This completes the derivation.

3.7.4 Completeness of the construction

What we have shown is how one can construct (anti-)self-dual potentials from the matrices a and b. **To prove that all the solutions can be obtained this way, one must do the following:**

- **Given a (A)SD potential A_μ, construct, up to gauge transformation, the matrices a and b.**
- **Show further that such a, b reproduce the potential A_μ one started with.**

Corrigan et al have shown that this can be achieved by making use of the solutions of the Dirac equation in the presence of A_μ.

□ **Sketch of the proof**:

Since the proof is rather long and technical, we shall only sketch the logic of the procedure.

Massless Dirac fields are quite sensitive to the presence of the instanton and carries the essential information about the instanton field itself.

Let A_μ be the **self-dual** instanton with Pontryagin number k for the gauge group $SU(n)$ and consider the Dirac equation for the fermion in the fundamental representation of $SU(n)$. It is given by

$$\not{D}\Psi = \gamma_\mu D_\mu \Psi = 0\,, \qquad D_\mu = \partial_\mu + A_\mu \tag{3.34}$$

$$\gamma_\mu = \begin{pmatrix} 0 & e_\mu \\ e_\mu^\dagger & 0 \end{pmatrix}, \qquad \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu} \tag{3.35}$$

Make the chiral decomposition of Ψ:

$$\gamma_5 \equiv \gamma_1\gamma_2\gamma_3\gamma_4 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.36}$$

$$\Psi = \begin{pmatrix} \psi^- \\ \psi^+ \end{pmatrix}, \qquad \gamma_5\psi^\mp = \mp\psi^\mp \tag{3.37}$$

Then the Dirac equation splits into

$$e_\mu D_\mu \psi^+ = 0\,, \qquad e_\mu^\dagger D_\mu \psi^- = 0 \tag{3.38}$$

Important properties of the solutions[1]

1. There are **no normalizable solutions for ψ^+**.
2. There are exactly **k normalizable solutions for ψ^-** (in the fundamental representation).

To see property 1, act $e_\mu^\dagger D_\mu$ to the ψ^+ equation. Using $e_\mu^\dagger e_\nu = \delta_{\mu\nu} + i\bar{\eta}_{\mu\nu}$, (where $\bar{\eta}_{\mu\nu} \equiv 2\bar{\sigma}_{\mu\nu}$), we have

$$\begin{aligned} 0 = e_\mu^\dagger D_\mu e_\nu D_\nu \psi^+ &= D^2\psi^+ + \frac{i}{2}\bar{\eta}_{\mu\nu}[D_\mu, D_\nu]\psi^+ \\ &= D^2\psi^+ + \frac{i}{2}\underbrace{\bar{\eta}_{\mu\nu}F_{\mu\nu}}_{0}\psi^+ = D^2\psi^+ \end{aligned} \tag{3.39}$$

[1] Brown et al (PRD16(77)417

But since $-D^2$ is a positive definite operator,

$$
\begin{aligned}
0 &= \int d^4\psi^{+*}(-D)^2\psi^+ = \int d^4x|D_\mu\psi^+|^2 \\
&\Rightarrow \quad D_\mu\psi^+ = 0 \\
&\Rightarrow 0 = [D_\mu, D_\nu]\psi^+ = F_{\mu\nu}\psi^+ \\
&\Rightarrow \psi^+ = 0
\end{aligned}
$$

Property 2 follows from the Atiyah-Singer index theorem for the Dirac operator:

$$n_+ - n_- = -\frac{1}{32\pi^2}\int d^4F^a_{\mu\nu}\tilde{F}^a_{\mu\nu} = -k \tag{3.40}$$

$$n^\pm = \text{number of normalizable solutions for } \psi^\pm. \tag{3.41}$$

Since $n_+ = 0$ as shown above, $n_- = k$.

Infact, the explicit solution for ψ^- can be expressed in terms of the ADHM data as [2] (hereafter we will write ψ^- as ψ)

$$\boldsymbol{\tilde{\psi} = \frac{1}{\pi} v^\dagger b f} \quad (= n \times 2k)\,, \qquad (\tilde{\psi}_A \equiv \epsilon_{AB}\psi_B) \tag{3.42}$$

$$v^\dagger = n \times (2k+n)\,, \quad b = (2k+n)\times 2k\,, \quad f = 2k \times 2k \tag{3.43}$$

Hereafter, it is convenient to use **the canonical form for *a* and *b***, *i.e.*

$$a = \begin{pmatrix} \lambda \\ \mu \end{pmatrix}, \qquad \lambda = n \times 2k\,, \quad \mu = 2k \times 2k \tag{3.44}$$

$$b = \begin{pmatrix} 0_{n\times 2k} \\ 1_{2k} \end{pmatrix} \tag{3.45}$$

where λ and μ are quaternionic.

One now proceeds in the following way:

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[2]Gorrigan et al, NPB140(78) 31, Osborn, NPB140 (78) 45

1. Prove that ψ is normalizable. This is done by the use of the relation

$$\psi^\dagger \psi = -\frac{1}{4\pi^2}\partial^2 f \tag{3.46}$$

$$f^{-1} = a^\dagger a + 2\mu_\mu x_\mu + x^2 \tag{3.47}$$

Using integration by parts, one verifies the normalizability

$$\int d^4 \psi^\dagger \psi = 1 \tag{3.48}$$

2. By taking the first moment, one obtains

$$(\star) \qquad \mu_\mu = -\int d^4 x x_\mu \psi^\dagger \psi \tag{3.49}$$

This gives half of a.

3. To get the remaining half of a, *i.e.* λ, one looks at the large x behavior of the solution $\tilde{\psi}$. The result is

$$\tilde{\psi} \sim -\frac{g\lambda x^\dagger}{x^4}, \qquad |x| \to \infty \tag{3.50}$$

where g is the gauge transformation defined by the asymptotic behavior of A_μ as

$$A_\mu \sim (\partial_\mu g) g^{-1}, \qquad |x| \to \infty \tag{3.51}$$

At this point, *a* and *b* have been completely determined.

4. One must show that **such data have the required properties for the ADHM construction**.

First, one needs to prove that $a^\dagger a = \mu^\dagger \mu + \lambda^\dagger \lambda$ commutes with the quaternions, *i.e.* proportional to e_0. By using $(\star)$ above for μ_ν, one considers

$$\mu_\mu \mu_\nu = \int\int d^4 x d^4 y x_\mu y_\nu \psi^\dagger(x) \boldsymbol{\psi(x)\psi^\dagger(y)} \psi(y) \tag{3.52}$$

The information about $\psi(x)\psi^\dagger(y)$ is carried by the Green's function in the presence of the instanton.

$$D^2 G(x,y) = -\delta(x-y) \tag{3.53}$$

$$e_\mu e_\mu^\dagger D_\mu G(x,y)\overleftarrow{D}_\nu = -\delta(x-y) + \boldsymbol{\psi(x)\psi^\dagger(y)} \tag{3.54}$$

Making use of the formula, one can show that indeed $a^\dagger a$ commutes with the quaternions.

5. Remaining steps:

   - Show that $v(x)$ satisfying the relation $v^\dagger \Delta = 0$ exists.
   - v is appropriately normalized: $v^\dagger v = 1$.
   - Show that $\Delta^\dagger(x)\Delta(x)$ is non-sigular everywhere.

In this way, the completeness of the ADHM construction was established in a "physical" way.