(1930s)

Hodge theory for Riemannian Manifolds

(An extension of de Rham cohomology 1920s)

Let M be a closed smooth manifold, the de Rham complex

$$0 \to \Omega^0(M) \xrightarrow{d_0} \Omega^1(M) \xrightarrow{d_1} \cdots \xrightarrow{d_{n-1}} \Omega^n(M) \xrightarrow{d_n} 0 .$$

de Rham theorem : $\quad H^k(M,\mathbb{R}) \cong \dfrac{\ker d_k}{\operatorname{Im} d_{k-1}}$.

(A) Hodge duality : $\quad$ P-forms in $\mathbb{R}^2, \mathbb{R}^3, \mathbb{R}^4$

	Basic P-forms	dimensionality
	1	1
$\mathbb{R}^2$	dx, dy	2
	$dx\wedge dy$	1
	1	1
	dx, dy, dz	3
$\mathbb{R}^3$	$dx\wedge dy, dy\wedge dz, dz\wedge dx$	3
	$dx\wedge dy\wedge dz$	1
		1
		4
$\mathbb{R}^4$		6
		4
		1

"$*$" ← Hodge star operator ,

Observed that

$$\# \text{ of } p\text{-form} = \# \text{ of } (m-p)\text{-form} ,$$

$$*(p\text{-form}) = (m-p)\text{-form} .$$

$\boxed{\text{Def}}$

$$*\left(dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_r} \right) = \frac{1}{(m-r)!} \, \epsilon^{\mu_1 \mu_2 \cdots \mu_r}{}_{\nu_{r+1} \cdots \nu_m} \, dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_m}$$

Note that : "$*$" 與 metric 有關 in G.R. (curved spacetime),

$\because$ $\epsilon^{\mu\nu\alpha\beta}$ (例) 與 metric 有關

是一個 tensity density. (<u>NOT tensor</u>).

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Remember

$$\epsilon_{\mu_1 \mu_2 \cdots \mu_m} = \begin{cases} +1 & \text{even permutation of } (1, 2, \cdots m) \\ -1 & \text{odd permutation of } (1, 2, \cdots m) \\ 0 & \text{otherwise.} \end{cases} \quad \text{for } \underline{\underline{\text{All}}} \text{ coordinates}$$

)

For $T_{\mu\nu\alpha\beta} \equiv T_{[\mu\nu\alpha\beta]}$, a totally antisymmetric tensor ,

$$\begin{cases} T'_{0123} = \dfrac{\partial x^{\kappa}}{\partial x'^{0}} \dfrac{\partial x^{\lambda}}{\partial x'^{1}} \dfrac{\partial x^{\mu}}{\partial x'^{2}} \dfrac{\partial x^{\nu}}{\partial x'^{3}} \, T_{\kappa\lambda\mu\nu} = D \, T_{0123} . \\ \\ \text{But} \quad \epsilon'_{\mu\nu\alpha\beta} = \epsilon_{\mu\nu\alpha\beta} \quad \text{by definition !!} \end{cases}$$

其中

$$D \equiv \begin{vmatrix} \dfrac{\partial x^0}{\partial x'^0} & \dfrac{\partial x^1}{\partial x'^1} & \dfrac{\partial x^2}{\partial x'^2} & \dfrac{\partial x^3}{\partial x'^3} \\ \dfrac{\partial x^1}{\partial x'^0} & & & \\ \vdots & & & \\ \dfrac{\partial x^3}{\partial x'^0} & & & \dfrac{\partial x^3}{\partial x'^3} \end{vmatrix} \equiv \det \frac{\partial x^{\mu}}{\partial x'^{\nu}} .$$

- Tensor density in GR.

tensor density of weight w （以 $\binom{1}{1}$ tensor 為例）:

Def : $$S'^{\mu}{}_{\nu} = D^{w} \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x^{\sigma}}{\partial x'^{\nu}} S^{\rho}{}_{\sigma} .$$

例 : ① $$\epsilon'^{\mu\nu\alpha\beta} = D \frac{\partial x'^{\mu}}{\partial x^{\rho}} \frac{\partial x'^{\nu}}{\partial x^{\sigma}} \frac{\partial x'^{\alpha}}{\partial x^{\tau}} \frac{\partial x'^{\beta}}{\partial x^{\xi}} \epsilon^{\rho\sigma\tau\xi}$$

$$= D \cdot \underbrace{\det \frac{\partial x'}{\partial x}}_{=\frac{1}{D}} \cdot \epsilon^{\mu\nu\alpha\beta} = \epsilon^{\mu\nu\alpha\beta} ,$$

$\therefore$ $\epsilon^{\mu\nu\alpha\beta}$: tensor density of weight 1.

② $g \equiv \det g_{\mu\nu}$ is a tensor density.

$$g'_{\mu'\nu'} = \frac{\partial x^{\rho}}{\partial x^{\mu'}} \frac{\partial x^{\sigma}}{\partial x^{\nu'}} g_{\rho\sigma} \rightarrow g' = D^{2} g ,$$

$$\therefore \sqrt{g'} = D \sqrt{g} .$$

③ The volume element is a tensor density.

$$d^{m}x = dx^{0} \wedge dx^{1} \wedge \cdots \wedge dx^{m-1} = \frac{1}{m!} \epsilon_{\mu_1\mu_2\cdots\mu_m} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_m} ,$$

but $$\underbrace{\epsilon_{\mu_1\mu_2\cdots\mu_m}}\, dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_m} = \underbrace{\epsilon_{\mu_1'\mu_2'\cdots\mu_m'}} \frac{\partial x^{\mu_1}}{\partial x^{\mu_1'}} \cdots \frac{\partial x^{\mu_m}}{\partial x^{\mu_m'}} dx^{\mu_1'} \wedge \cdots \wedge dx^{\mu_m'}$$

相等 by def

$$= \underbrace{\left| \frac{\partial x^{\mu}}{\partial x^{\mu'}} \right|}_{=D} \epsilon_{\mu_1'\cdots\mu_m'} dx^{\mu_1'} \wedge \cdots \wedge dx^{\mu_m'}$$

∴ invariant volumn element is defined to be

$$\boxed{\sqrt{g}\, d^m x = \sqrt{g}\, dx^0 \wedge \cdots dx^m = \frac{\sqrt{g}}{m!} \epsilon_{\mu_1 \cdots \mu_m} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_m} .}$$

Def

In curved manifold, "$*$" is defined to be

$$*\left(dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r} \right) = \frac{\sqrt{g}}{(m-r)!} \epsilon^{\mu_1 \mu_2 \cdots \mu_r}{}_{\nu_{r+1} \cdots \nu_m} dx^{\nu_{r+1}} \wedge \cdots \wedge dx^{\nu_m} .$$

Note that

① $$*1 = \frac{\sqrt{g}}{m!} \epsilon_{\nu_1 \nu_2 \cdots \nu_m} dx^{\nu_1} \wedge \cdots dx^{\nu_m} = \sqrt{g}\, dx^0 \wedge \cdots \wedge dx^m$$

= invariant volumn element .

② For Lorentzian signature, $\sqrt{g} \to \sqrt{|g|}$.

③ For non-coordinate basis $\{ \hat{\theta}^\alpha \} = \{ e^\alpha{}_\mu dx^\mu \}$, $g \equiv 1$

↖ vielbein

$$*\left(\hat{\theta}^{\alpha_1} \wedge \cdots \wedge \hat{\theta}^{\alpha_r} \right) = \frac{1}{(m-r)!} \epsilon^{\alpha_1 \cdots \alpha_r}{}_{\beta_{r+1} \cdots \beta_m} \hat{\theta}^{\beta_{r+1}} \wedge \cdots \wedge \hat{\theta}^{\beta_m} .$$

For
$$\omega = \frac{1}{r!}\,\omega_{\mu_1\mu_2\cdots\mu_r}\, dx^{\mu_1}\wedge dx^{\mu_2}\wedge\cdots\wedge dx^{\mu_r} \in \Omega^r(M),$$

we have
$$*\omega = \frac{\sqrt{|g|}}{r!\,(m-r)!}\,\omega_{\mu_1\cdots\mu_r}\,\epsilon^{\mu_1\cdots\mu_r}{}_{\nu_{r+1}\cdots\nu_m}\, dx^{\nu_{r+1}}\wedge\cdots\wedge dx^{\nu_m}.$$

Theorem : Let $\omega \in \Omega^r(M)$, Then

$$**\omega = (-1)^{r(m-r)}\cdot\omega.$$

proof : (For the case of non-coordinate basis.)

$$**\omega = \frac{1}{r!}\,\omega_{\alpha_1\cdots\alpha_r}\,\frac{1}{(m-r)!}\,\epsilon^{\alpha_1\cdots\alpha_r}{}_{\beta_{r+1}\cdots\beta_m} \times \frac{1}{[m-(m-r)]!}\,\epsilon^{\beta_{r+1}\cdots\beta_m}{}_{\gamma_1\cdots\gamma_r}\,\hat\theta^{\gamma_1}\wedge\cdots\wedge\hat\theta^{\gamma_r}$$

$$= \frac{(-1)^{r(m-r)}}{r!\,r!\,(m-r)!}\sum_{\alpha\beta\gamma}\omega_{\alpha_1\cdots\alpha_r}\,\epsilon_{\alpha_1\cdots\alpha_r\beta_{r+1}\cdots\beta_m}\,\epsilon_{\gamma_1\cdots\gamma_r\beta_{r+1}\cdots\beta_m}\,\hat\theta^{\gamma_1}\wedge\cdots\wedge\hat\theta^{\gamma_r}$$

use
$$\sum_{\beta\gamma}\epsilon_{\alpha_1\cdots\alpha_r\beta_{r+1}\cdots\beta_m}\,\epsilon_{\gamma_1\cdots\gamma_r\beta_{r+1}\cdots\beta_m}\,\hat\theta^{\gamma_1}\wedge\cdots\wedge\hat\theta^{\gamma_r} = r!\,(m-r)!\,\hat\theta^{\alpha_1}\wedge\cdots\wedge\hat\theta^{\alpha_r} \quad (*)$$

(驗証) $m=3$, $r=2$. (see Marion Chap I).

$$\sum_{\beta\gamma}\epsilon_{\alpha_1\alpha_2\beta}\,\epsilon_{\gamma_1\gamma_2\beta}\,\hat\theta^{\gamma_1}\wedge\hat\theta^{\gamma_2} = \sum_{\gamma}\left(\delta_{\alpha_1\gamma_1}\delta_{\alpha_2\gamma_2} - \delta_{\alpha_1\gamma_2}\delta_{\alpha_2\gamma_1}\right)\hat\theta^{\gamma_1}\wedge\hat\theta^{\gamma_2}$$

$$= \hat\theta^{\alpha_1}\wedge\hat\theta^{\alpha_2} - \hat\theta^{\alpha_2}\wedge\hat\theta^{\alpha_1} = 2\,\hat\theta^{\alpha_1}\wedge\hat\theta^{\alpha_2} = 2!\,(3-2)! \times \hat\theta^{\alpha_1}\wedge\hat\theta^{\alpha_2}$$

Ok!!

$$= \frac{(-1)^{r(m-r)}}{r!}\,\omega_{\alpha_1\cdots\alpha_r}\,\hat\theta^{\alpha_1}\wedge\cdots\wedge\hat\theta^{\alpha_r} = (-1)^{r(m-r)}\,\omega$$

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So $(-1)^{r(m-r)} ** \omega = \omega$

$\Rightarrow$ $(-1)^{r(m-r)} **$ is an identity map on $\Omega^r(M)$,

$\Rightarrow$ $*^{-1} = (-1)^{r(m-r)} *$

(B) <u>Inner products of r-forms.</u>

To study cohomology of M:

$g_{\mu\nu} \longrightarrow * \longrightarrow (\ ,\ ) \longrightarrow d^{\dagger} \longrightarrow$ Harmonic forms.

(metric on M) (Hodge star) (inner product on $\Omega^r(M)$) (adjoint) (Laplacian)

Take

$$\omega = \frac{1}{r!} \omega_{\mu_1 \cdots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}, \quad \eta = \frac{1}{r!} \eta_{\mu_1 \cdots \mu_r} dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}$$

$$\omega \wedge * \eta = \frac{1}{(r!)^2} \omega_{\mu_1 \cdots \mu_r} \eta_{\nu_1 \cdots \nu_r} \frac{\sqrt{g}}{(m-r)!} \epsilon^{\nu_1 \cdots \nu_r}{}_{\mu_{r+1} \cdots \mu_m} \underbrace{dx^{\mu_1} \wedge \cdots dx^{\mu_r} \wedge dx^{\mu_{r+1}} \wedge \cdots \wedge dx^{\mu_m}}$$

$$= \frac{1}{r!} \sum_{\mu,\nu} \omega_{\mu_1 \cdots \mu_r} \eta^{\nu_1 \cdots \nu_r} \frac{\sqrt{g}}{r!(m-r)!} \epsilon_{\nu_1 \cdots \nu_r \mu_{r+1} \cdots \mu_m} \underbrace{\epsilon^{\mu_1 \cdots \mu_r \mu_{r+1} \cdots \mu_m} \cdot dx^1 \wedge \cdots \wedge dx^m}$$

$$\overset{用 ⊛ P.5}{=} \frac{1}{r!} \omega_{\mu_1 \cdots \mu_r} \eta^{\mu_1 \cdots \mu_r} \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^m$$

$$\therefore \quad \omega \wedge {*\eta} = \eta \wedge {*\omega} \quad , \quad \text{Symmetric !}$$

$\boxed{\text{Def}}$ inner product

$$(\omega, \eta) \equiv \int \omega \wedge *\eta = \frac{1}{r!} \int_M \omega_{\mu_1 \cdots \mu_r} \eta^{\mu_1 \cdots \mu_r} \underbrace{\sqrt{g}\, dx^1 \wedge \cdots \wedge dx^m}$$

$$(\omega, \omega) \geq 0, \quad \text{if } (M, g) \text{ is Riemannian.}$$

(c) Adjoints of exterior derivatives

$$d : \Omega^{r-1}(M) \to \Omega^r(M) \qquad , \quad \dim M = m$$

Def : $d^\dagger : \Omega^r(M) \to \Omega^{r-1}(M)$

$$d^\dagger = (-1)^{mr+m+1} * d *$$

diagram :

$$\begin{tikzcd}
\Omega^{m-r}(M) \arrow[r, "(-1)^{mr+m+1} d"] & \Omega^{m-r+1}(M) \arrow[d, "*"] \\
\Omega^r(M) \arrow[u, "*"] \arrow[r, "d^\dagger"] & \Omega^{r-1}(M)
\end{tikzcd}$$

Note that $d^\dagger$ is nilpotent : $\quad d^{\dagger 2} = *d**d* \propto *d^2* = 0$

Theorem : Let (M, g) be a compact orientable manifold without boundary and $\alpha \in \Omega^r(M)$, $\beta \in \Omega^{r-1}(M)$. Then

$$(d\beta, \alpha) = (\beta, d^\dagger \alpha)$$

proof : Note that $d\beta \wedge * \alpha$ and $\beta \wedge * d^\dagger \alpha$ are m-forms.

consider $d(\beta \wedge * \alpha) = d\beta \wedge * \alpha - (-1)^r \beta \wedge d * \alpha$

The identity map

$$(-1)^{r(m-r)} ** = (-1)^{(m-r+1)[m-(m-r+1)]} ** = (-1)^{mr+m+r+1} **$$

$$\therefore \quad d(\beta \wedge * \alpha) = d\beta \wedge * \alpha - (-1)^{mr+m+1} \beta \wedge *(* d * \alpha)$$

$$\Rightarrow \int_M d\beta \wedge * \alpha - \int_M \beta \wedge * \underbrace{[(-1)^{mr+m+1} * d * \alpha]}_{d^\dagger \alpha} = \int_M d(\beta \wedge * \alpha)$$

$$= \int_{\partial M} \beta \wedge * \alpha = 0$$

$$\therefore \quad (d\beta, \alpha) = (\beta, d^\dagger \alpha)$$

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(D) The Laplacian, harmonic forms and the Hodge decomposition theorem

$\boxed{\text{Def}}$: The Laplacian $\Delta : \Omega^r(M) \to \Omega^r(M)$ is defined by

$$\Delta = (d + d^\dagger)^2 = dd^\dagger + d^\dagger d$$

Def : An r-form ω

① is called harmonic if $\Delta\omega = 0$,

② is called closed if $d\omega = 0$,

③ is called co-closed if $d^\dagger\omega = 0$.

$\boxed{\text{Theorem}}$ An r-form ω is harmonic $\Longleftrightarrow$ ω is closed and co-closed.

proof : $(\omega, \Delta\omega) = (\omega, (d^\dagger d + dd^\dagger)\omega)$

$= (d\omega, d\omega) + (d^\dagger\omega, d^\dagger\omega) \geq 0$

$\boxed{\text{Theorem}}$ Hodge decomposition theorem

Let (M, g), a compact, orientable Riemannian manifold without boundary.
(positive definite metric)

Then $\Omega^r(M)$ is uniquely decomposed as

$$\Omega^r(M) = d\Omega^{r-1}(M) \oplus d^\dagger\Omega^{r+1}(M) \oplus \mathrm{Harm}^r(M)$$

or

$$\boxed{\omega_r = d\alpha_{r-1} + d^\dagger\beta_{r+1} + \gamma_r} \quad \text{(globally)}$$

proof : The existence proof is highly technical !!

We will only give the uniqueness proof.

$$\underset{\text{uniqueness}}{} \qquad \overset{\text{p-form}}{\omega} = \overset{\text{harmonic part}}{\alpha} + \underset{\text{exact part}}{d\beta} + \overset{\text{co-exact part}}{d^\dagger\gamma}$$

(Candelas)

$\boxed{\text{Note}}$ $\begin{cases} (\alpha, d\beta) = 0 \\ (d\beta, d^\dagger\gamma) = 0 \\ (\alpha, d^\dagger\gamma) = 0 \end{cases}$ (*)

即證明：若 $0 = \alpha + d\beta + d^\dagger\gamma \Rightarrow \alpha = 0,\ d\beta = 0,\ d^\dagger\gamma = 0$

作用 d $\Rightarrow$ $0 = \cancelto{0}{d\alpha} + \cancelto{0}{d^2\beta} + dd^\dagger\gamma$

(see p.9)

$\therefore\ dd^\dagger\gamma = 0 \Rightarrow (\gamma, dd^\dagger\gamma) = 0 \Rightarrow (d^\dagger\gamma, d^\dagger\gamma) = 0 \Rightarrow d^\dagger\gamma = 0$

作用 $d^\dagger$ $\Rightarrow$ $0 = \cancelto{0}{d^\dagger\alpha} + d^\dagger d\beta + \cancelto{0}{(d^\dagger)^2\gamma}$

$\therefore\ d\beta = 0$

$\Rightarrow\ \alpha = 0$ 始.

An important observation :

$\boxed{\text{Theorem}}$ Let ω be a closed form. ⊠

$$\Rightarrow \underset{\text{closed form}}{\omega} = \underset{\substack{\text{harmonic}\\\text{part}}}{\alpha} + \overset{\text{exact part}}{d\beta}$$

proof :

$$\omega = \alpha + d\beta + d^\dagger\gamma$$

$$\cancelto{0}{d\omega} = \cancelto{0}{d\alpha} + \cancelto{0}{d^2\beta} + dd^\dagger\gamma$$

$\therefore\ dd^\dagger\gamma = 0 \Rightarrow d^\dagger\gamma = 0$

$$\boxed{\underset{\text{closed form}}{\omega} = \underset{\text{harmonic part}}{\alpha} + \underset{\text{exact part}}{d\beta}}$$

An important application ;

Question : when is a closed form exact ?

since a harmonic form is Never exact $\overbrace{}^{(\text{except } \alpha = 0,\ \text{see } (*) \text{ in p.10})}$ this is equivalent to

asking $\boxed{\text{Question}}$: When a closed form has <u>nonzero harmonic part</u> ?

Thus, the existence of <u>harmonic forms</u> is related to the

<u>global properties</u> of the manifold.

$\boxed{[13]}$ $M = T_2$ Harmonic 1-forms for two-Torus.

(Candelas)

arbitrary one-form on T_2

$\omega = u(x,y)\,dx + v(x,y)\,dy$. (Note that "$dx$", "$dy$" are NOT exact!)

perodic B.C. $\Rightarrow$ $\omega = \sum_{m,n=0}^{\infty} (u_{mn}\,dx + v_{mn}\,dy)\,e^{i(mx+ny)}$

<u>Hodge decomposition</u> :

$$\omega = u_{00}\,dx + v_{00}\,dy + d\left\{-i\sum{}' \frac{m u_{mn} + n v_{mn}}{m^2+n^2}\, e^{i(mx+ny)}\right\}$$
$$+ d^{\dagger}\left\{-i\sum{}' \frac{n u_{mn} - m v_{mn}}{m^2+n^2}\, e^{i(mx+ny)}\, dx\wedge dy\right\}$$

two linearly independent harmonic 1-forms on T_2 $\Rightarrow b_1 = 2$

An undergrad mathematical physics theorem

$\boxed{\text{Theorem}}$ The Helmholtz's Theorem in vector analysis in R^3

a vector field $\vec{a}$ can be decomposed into

$$\vec{a} = -\vec{\nabla}\varphi + \vec{\nabla}\times\vec{A}$$

vector potential

scalar potential

proof :

consider the identity

$$\nabla^2\vec{v} = \vec{\nabla}\,\nabla\cdot\vec{v} - \vec{\nabla}\times(\nabla\times\vec{v}) \quad \cdots\cdots (1)$$

Define
$$\vec{v}(\vec{r}) = -\frac{1}{4\pi}\int_{R^3}\frac{\vec{a}}{|\vec{r}-\vec{r}'|}\,d\vec{r}' \quad \cdots (2)$$

assume $\vec{a}\to 0$ faster than $\frac{1}{r^2}$.

Then
$$\nabla_r^2 v_i = -\frac{1}{4\pi}\int_{R^3} a_i(\vec{r}')\,\nabla_r^2\frac{1}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$
$$= -\frac{1}{4\pi}\int_{R^3} a_i(\vec{r}')\left[-4\pi\,\delta(\vec{r}-\vec{r}')\right]d\vec{r}'$$
$$= a_i(\vec{r}).$$

$\therefore$ (1) gives $\vec{a} = -\vec{\nabla}\varphi + \nabla\times\vec{A}$

with $\varphi = -\vec{\nabla}\cdot\vec{v}$, $\vec{A} = -\nabla\times\vec{v}$

where $\vec{v}$ is defined in (2) $\quad$ ※

(E) Harmonic forms and de Rham cohomology groups.

We can show that any element of the de Rham cohomology group has a unique harmonic representative.

By ⊠ in P. 10

①

$$\underset{\text{closed form}}{\omega} = \underset{\text{harmonic part}}{\alpha} + \underset{\text{exact part}}{d\beta}.$$

② uniqueness theorem in P. 10

若 $\omega = \alpha_1 + d\beta_1 = \alpha_2 + d\beta_2 \Rightarrow \alpha_1 - \alpha_2 = d(\beta_1 - \beta_2)$

$\Rightarrow \alpha_1 - \alpha_2$ is harmonic & closed $\Rightarrow \alpha_1 - \alpha_2 = 0$ or $\alpha_1 = \alpha_2$

③ 若 $\omega_1 = \alpha + d\beta_1$, $\omega_2 = \alpha + d\beta_2$ $\Rightarrow \omega_1 - \omega_2 = d(\beta_1 - \beta_2)$

$\Rightarrow$ ω_1 and ω_2 belong to the same de Rham cohomology class. 证

Theorem Hodge theorem

$$H^r(M) \cong \mathrm{Harm}^r(M)$$

In particular.

$$\dim \mathrm{Harm}^r(M) = \dim H^r(M) = b^r$$

the Euler characteristic

$$\chi(M) = \sum (-1)^r b^r = \sum (-1)^r \dim \mathrm{Harm}^r(M)$$

topological quantity

analytical quantity given by the eigenvalue problem of the Laplacian Δ.