1

Integration of Diff forms on Manifolds

- "line integral" on R^3

$\vec{\alpha}(t) = (x(t), y(t), z(t))$

$$\int_C \vec{F}\cdot d\vec{\alpha} = \int_{t_1}^{t_2} \vec{F}(\vec{\alpha}(t))\cdot\vec{\alpha}'(t)\,dt = \int_{t_1}^{t_2} F_1[\vec{\alpha}(t)]\frac{dx}{dt}dt + F_2[\vec{\alpha}(t)]\frac{dy}{dt}dt + F_3[\vec{\alpha}(t)]\frac{dz}{dt}dt$$

or $$= \int_C F_1(x,y,z)\,dx + F_2(x,y,z)\,dy + F_3(x,y,z)\,dz \quad (\text{1-form})$$

(indep. of parametrization)

例) $\vec{F}(x,y) = \sqrt{y}\,\vec{i} + (x^3+y)\,\vec{j}$ (Apostol P 325)

ⓐ $\vec{\alpha}(t) = (t, t)$, $0 \le t \le 1$

ⓑ $\vec{\alpha}(t) = (t^2, t^3)$, $0 \le t \le 1$

Sol: ⓐ $\vec{\alpha}'(t) = (1,1)$, $\vec{F}(\vec{\alpha}(t)) = (\sqrt{t}, t^3+t)$

$$\int \vec{F}\cdot d\vec{\alpha} = \int_0^1 (\sqrt{t} + t^3 + t)\,dt = \frac{17}{12}$$

ⓑ $\vec{\alpha}'(t) = (2t, 3t^2)$, $\vec{F}(\vec{\alpha}(t)) = (t^{3/2}, t^6+t^3)$

$$\int \vec{F}\cdot d\vec{\alpha} = \int_0^1 (2t^{5/2} + 3t^8 + 3t^5)\,dt = \frac{59}{42}$$

ⓑ' $\vec{\beta}(t) = (t, t^{3/2})$, $0 \le t \le 1$

$$\int \vec{F}\cdot d\vec{\beta} = \int_0^1 \left(t^{3/4} + \frac{3}{2}t^{7/2} + \frac{3}{2}t^2\right)dt = \frac{59}{42}$$ same with ⓑ!

indep of parametrization

Arc length

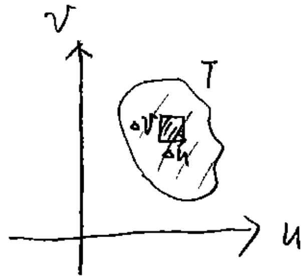
$$\text{Length}(C) = \int_{t_1}^{t_2} \| \vec{\alpha}'(t) \| \, dt$$

$$= \int_{t_1}^{t_2} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \; dt$$

$$= \int_C \sqrt{dx^2 + dy^2 + dz^2} \qquad (\; dx \text{ etc. } 1\text{-form} \;)$$

Surface Area

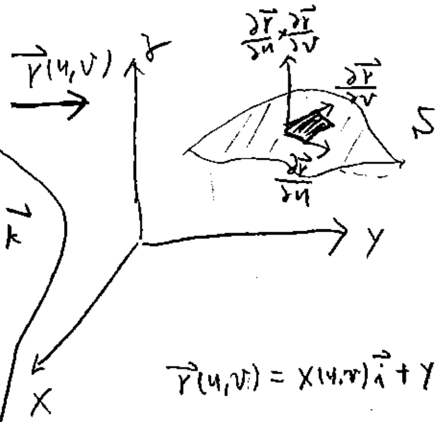
$$\underset{\displaystyle \overset{\shortparallel}{\int_S ds}}{\text{Area}(S)} = \int_T \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\, dv$$



$$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$$

$$\therefore \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} = \begin{vmatrix} \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \end{vmatrix} \vec{i} + \begin{vmatrix} \frac{\partial z}{\partial u} & \frac{\partial x}{\partial u} \\ \frac{\partial z}{\partial v} & \frac{\partial x}{\partial v} \end{vmatrix} \vec{j} + \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{vmatrix} \vec{k}$$

$$= \frac{\partial(y,z)}{\partial(u,v)} \vec{i} + \frac{\partial(z,x)}{\partial(u,v)} \vec{k} + \frac{\partial(x,y)}{\partial(u,v)} \vec{k}$$



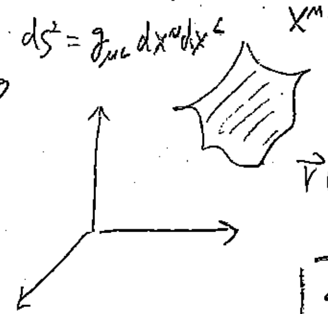
$\vec{r}(u,v) = x(u,v)\vec{i} + y(u,v)\vec{j} + z(u,v)\vec{k}$

$(u,v) \in T$

$$= \iint_T \sqrt{\left(\frac{\partial(y,z)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(z,x)}{\partial(u,v)}\right)^2 + \left(\frac{\partial(x,y)}{\partial(u,v)}\right)^2} \; du\,dv \;=\; \underset{(\text{ Invariant volume element })}{\iint \sqrt{g}\; du\,dv}$$

$$= \iint_S \sqrt{(dy \wedge dz)^2 + (dz \wedge dx)^2 + (dx \wedge dy)^2} \qquad \left(\begin{array}{l} dx \wedge dy \text{ etc.} \\ 2\text{-forms !!!} \end{array}\right)$$

$\boxed{\text{Intrinsic or Not ?}}$



the 1st fundamental form (Intrinsic)

$\boxed{4D\text{-}3}$

3d only!?

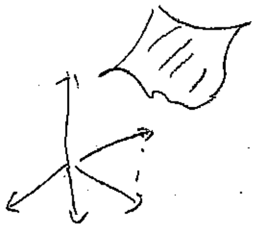
- $dA = |\vec{r}_u \times \vec{r}_v|\,du\,dv$

$$= \sqrt{|\vec{r}_u|^2 |\vec{r}_v|^2 - (\vec{r}_u \cdot \vec{r}_v)^2}\, du\,dv$$

$$= \sqrt{EG - F^2}\, du\,dv \quad (\text{Intrinsic}) = \sqrt{g}\, du\,dv$$

$$= \sqrt{(dx\wedge dy)^2 + (dy \wedge dz)^2 + (dz \wedge dx)^2}$$

- <u>string worldsheet</u>



$D = 26$

$$dA = \sqrt{EG - F^2}\, d\sigma\, d\tau \quad (\text{Intrinsic})$$

Nambu-Goto action.

- R^4 上の 3-volume.

(Intrinsic)
$ds^2 = g_{ij}\,dx^i dx^j$

$g_{ij} = \dfrac{\partial \vec{r}}{\partial u_i}\cdot\dfrac{\partial \vec{r}}{\partial u_j}$

$\vec{r} = (x_1, x_2, x_3, x_4)$

$$\sqrt{g}\, du_1 du_2 du_3 = \sqrt{(dx_1\wedge dx_2 \wedge dx_3)^2 + (dx_2 \wedge dx_3 \wedge dx_4)^2 + (dx_3 \wedge dx_4 \wedge dx_1)^2 + (dx_4 \wedge dx_1 \wedge dx_2)^2}\; du_1 du_2 du_3$$

(Intrinsic)

$$= \sqrt{\left|\frac{\partial(x_1\, x_2\, x_3)}{\partial(u_1\, u_2\, u_3)}\right|^2 + \left|\frac{\partial(x_2\, x_3\, x_4)}{\partial(u_1\, u_2\, u_3)}\right|^2 + \left|\frac{\partial(x_3\, x_4\, x_1)}{\partial(u_1\, u_2\, u_3)}\right|^2 + \left|\frac{\partial(x_4\, x_1\, x_2)}{\partial(u_1\, u_2\, u_3)}\right|^2}\; du_1 du_2 du_3$$

$$= \left\| \begin{vmatrix} \vec{i}_1 & \vec{i}_2 & \vec{i}_3 & \vec{i}_4 \\ \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} & \frac{\partial x_3}{\partial u_1} & \frac{\partial x_4}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} & \frac{\partial x_3}{\partial u_2} & \frac{\partial x_4}{\partial u_2} \\ \partial x_1 & \partial x_2 & \partial x_3 & \partial x_4 \end{vmatrix} \right\| du_1 du_2 du_3$$

• "Surface integral" on R^3

Note: 1. $dx \wedge dy = -dy \wedge dx$ etc.

* 2. The value of $dx \wedge dy$ is the area spanned by $\frac{\partial \vec{r}}{\partial u}, \frac{\partial \vec{r}}{\partial v}$ projected onto the $dxdy$ plane !!

$$\iint_S \vec{F} \cdot \vec{n}\, dS = \iint_T \vec{F}[\vec{r}(u,v)] \cdot \vec{n}(u,v) \left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\| du\,dv$$

$$= \iint_T \vec{F} \cdot \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}\, du\,dv$$

$$\left\{ \begin{aligned} &\vec{n} = \frac{\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}}{\left\| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right\|} \\ &\vec{n_2} = -\vec{n} \\ &\vec{F} = P(x,y,z)\vec{i} + Q(x,y,z)\vec{j} + R(x,y,z)\vec{k} \end{aligned} \right.$$

$$= \iint_T P[\vec{r}(u,v)] \frac{\partial(y,z)}{\partial(u,v)} du\,dv + Q[\vec{r}(u,v)] \frac{\partial(z,x)}{\partial(u,v)} du\,dv + R[\vec{r}(u,v)] \frac{\partial(x,y)}{\partial(u,v)} du\,dv$$

$$= \iint_S \underbrace{P(x,y,z)\, dy \wedge dz + Q(x,y,z)\, dz \wedge dx + R(x,y,z)\, dx \wedge dy}_{\text{2-forms}}$$

(indep of parametrization) #

$\boxed{\text{Note}}$ {Area form} 见 Suhubi P. 409.

当 $\vec{F} = (P, Q, R) = \vec{n} = \left(\frac{x}{(x^2+y^2+z^2)^{1/2}}, \frac{y}{(x^2+y^2+z^2)^{1/2}}, \frac{z}{(x^2+y^2+z^2)^{1/2}} \right)$ 時

上式 回到 面積 area form #

- "3-manifold integral" on R^4

There are 4 3-forms on R^4:

$$dx \wedge dy \wedge dz \,,\quad dy \wedge dz \wedge dw \,,\quad dz \wedge dw \wedge dx \,,\quad dw \wedge dx \wedge dy .$$

$$\int_M \omega = \int_M P(x,y,z,w)\, dx \wedge dy \wedge dz + Q(x,y,z,w)\, dy \wedge dz \wedge dw$$
$$+ R(x,y,z,w)\, dz \wedge dw \wedge dx + S(x,y,z,w)\, dw \wedge dx \wedge dy .$$

3-manifold M : $\vec{r}(a,b,c) = \big(x(a,b,c), y(a,b,c), z(a,b,c), w(a,b,c)\big)$

$$(a,b,c) \in A .$$

$$\int_M \omega = \iiint_A P[\vec{r}(a,b,c)] \frac{\partial(x,y,z)}{\partial(a,b,c)}\, da\,db\,dc + Q[\vec{r}(a,b,c)] \frac{\partial(y,z,w)}{\partial(a,b,c)}\, da\,db\,dc$$
$$+ R[\vec{r}(a,b,c)] \frac{\partial(z,w,x)}{\partial(a,b,c)}\, da\,db\,dc + S[\vec{r}(a,b,c)] \frac{\partial(w,x,y)}{\partial(a,b,c)}\, da\,db\,dc .$$

[31]

$$\omega = dy\wedge dz\wedge dw - dx\wedge dz\wedge dw - 2y\, dx\wedge dy\wedge dz$$

$$\vec{r}(a,b,c) = (a+b,\ a+c,\ bc,\ a^2),\quad (a,b,c) \in 0 \le a,b,c \le 1$$

Ans :

$$\int_M \omega = \iiint \left\{ \begin{vmatrix} \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} & \frac{\partial w}{\partial a} \\ \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} & \frac{\partial w}{\partial b} \\ \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} & \frac{\partial w}{\partial c} \end{vmatrix} da\,db\,dc - \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial z}{\partial a} & \frac{\partial w}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial z}{\partial b} & \frac{\partial w}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial z}{\partial c} & \frac{\partial w}{\partial c} \end{vmatrix} da\,db\,dc \right.$$

$$\left. - 2(a+c) \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a} \\ \frac{\partial x}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b} \\ \frac{\partial x}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c} \end{vmatrix} da\,db\,dc \right\}$$

$$= \int_0^1\!\!\int_0^1\!\!\int_0^1 \left\{ -2ac - 2ab - 2(a+c)(-c-b) \right\} da\,db\,dc$$

$$= \int_0^1\!\!\int_0^1\!\!\int_0^1 (2bc + 2c^2)\, da\,db\,dc = \frac{7}{6}$$

• More Examples

$\boxed{例}$ $\omega = (x^2+y^2)\,dx\wedge dy + z\,dy\wedge dz$ a 2-form on R^3

$M=S$: 圓柱 $x^2+y^2=1$ $0\le z\le 1$

Ans : ① $\vec{r} : R \to S$

$$\vec{r}(\theta,z) = (\cos\theta, \sin\theta, z)$$

$$R = \{(\theta,z) \mid 0\le\theta\le 2\pi,\ 0\le z\le 1\}$$

②
$$\frac{\partial\vec{r}}{\partial\theta} = (-\sin\theta, \cos\theta, 0)$$

$$\frac{\partial\vec{r}}{\partial z} = (0, 0, 1)$$

③ $\omega \to (\cos^2\theta+\sin^2\theta)\begin{vmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{vmatrix} d\theta dz + z\begin{vmatrix} \cos\theta & 0 \\ 0 & 1 \end{vmatrix} d\theta dz$

④ $\displaystyle\int_0^1\int_0^{2\pi} z\cos\theta\, d\theta\, dz = 0$

$\boxed{例}$ $\omega = z^2\, dx \wedge dy$ a 2-form on R^3

$M = S$: 上半單位圓

$$\vec{r}(r,t) = (r\cos t,\ r\sin t,\ \sqrt{1-r^2}), \quad 0 \le t \le 2\pi,\ 0 \le r \le 1$$

$$\int_M \omega = \iint_R (1-r^2) \begin{vmatrix} \cos t & \sin t \\ -r\sin t & r\cos t \end{vmatrix} dr\,dt$$

$$= \int_0^{2\pi}\int_0^1 (r - r^3)\, dr\, dt = \frac{\pi}{2}$$

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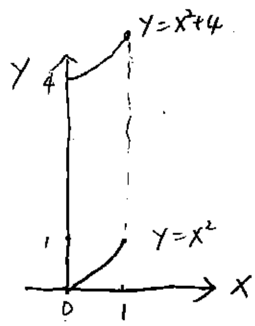
$\boxed{例}$ $\omega = (x^2+y)\, dx \wedge dy$ a 2-form on R^2

M :



設 $\vec{r}(u,v) = (u,\ u^2+v^2)$

$$\begin{cases} y = x^2 \Rightarrow v = 0 \\ y = x^2+4 \Rightarrow v = 2 \\ x = 0 \Rightarrow u = 0 \\ x = 1 \Rightarrow u = 1 \end{cases} \qquad \frac{\partial \vec{r}}{\partial u} = (1, 2u) \qquad \frac{\partial \vec{r}}{\partial v} = (0, 2v)$$

$$\therefore \int_M \omega = \int_0^2\int_0^1 \{u^2 + (u^2+v^2)\} \begin{vmatrix} 1 & 2u \\ 0 & 2v \end{vmatrix} du\,dv = \int_0^2\int_0^1 (4vu^2 + 2v^3)\, du\, dv$$

$$= \int_0^2 \left(\frac{4}{3}v + 2v^3\right) dv = \frac{32}{3}$$

\#

$\boxed{例}$ $\omega = x^2+z^2\, dx\wedge dy + (x^2+y^2)\, dy\wedge dz + (y^2+z^2)\, dz\wedge dx$ in R^3.

M = 2-chain $C_2 = \sum_{i=0}^{3} \sigma_2^i$

$$\begin{cases} \sigma_2^0 = (P_1, P_2, P_3)\ , \ \sigma_2^1 = -(P_0, P_2, P_3) \\ \sigma_2^2 = (P_0, P_1, P_3)\ , \ \sigma_2^3 = -(P_0, P_1, P_2) \end{cases}$$

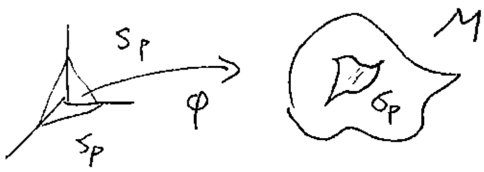
$C_2 = \partial C_3$
$= \sum_{i=0}^{3} (-)^i (P_0 \cdots \hat{P_i} \cdots P_3)$

$\boxed{\text{Def}}$ simplex & chain — the building blocks of a Manifold.

- $P_0, P_1, \ldots, P_p$ distinct points in R^n
- $\overline{P_0P_i}$ $(i=1,2,\cdots p)$ be linear independent or $S_2 = \{(u,v) \in R^2 : 0 \le u, v \le 1, u+v \le 1\}$
- A standard p-simplex $s_p = \left\{ P = \sum_{i=0}^{p} t_i P_i \ ,\ t_i \ge 0\ ,\ \sum_{i=0}^{p} t_i = 1 \right\}$
- Orientations of S_p $(P_1\, P_0\, P_2 \cdots P_p) = -S_p$ odd permutations

  $(P_0\, P_1\, P_2 \cdots P_p)$ even permutations

- A singular p-simplex in M



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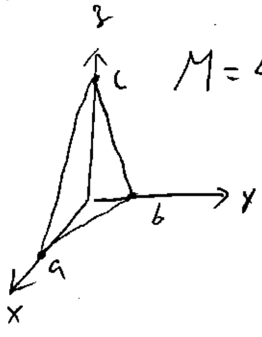
Now for M.

$P_0 = (0,0,0)\ ,\ P_1 = (a,0,0)\ ,\ P_2 = (0,b,0)\ ,\ P_3 = (0,0,c)$



$M = 4$ face

参数式 :

$$\varphi \equiv \begin{cases} \vec{r}_0(u,v) = (au, bv, c(1-u-v)) & \cdots \sigma_2^0 \\ \vec{r}_1(u,v) = (0, bu, cv) & \cdots \sigma_2^1 \\ \vec{r}_2(u,v) = (au, 0, cv) & \cdots \sigma_2^2 \\ \vec{r}_3(u,v) = (au, bv, 0) & \cdots \sigma_2^3 \end{cases}$$

Pull the form ω from faces back to standard simplex

$$\vec{r}_0^{\,*}\omega = \left[(au)^2 + c^2(1-u-v)^2\right]\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} du\,dv + \left[(au)^2 + (bv)^2\right]\begin{vmatrix} 0 & -c \\ b & -c \end{vmatrix} du\,dv$$

$$+\left[(bv)^2 + c^2(1-u-v)^2\right]\begin{vmatrix} -c & a \\ -c & 0 \end{vmatrix} du\,dv$$

$$= \left[a^2 b(a+c)u^2 + b^2 c(a+b)v^2 + ac^2(c+b)(1-u-v)^2\right] du\,dv$$

$$\vec{r}_1^{\,*}\omega = b^3 c\, u^2\, du\,dv$$

$$\vec{r}_2^{\,*}\omega = -ac^3 v^2\, du\,dv \qquad \text{pull-backs of } \omega \text{ by } \vec{r}_i$$

$$\vec{r}_3^{\,*}\omega = a^3 b\, u^2\, du\,dv$$

$$\Rightarrow \int_{\sigma_2^0} \omega = \int_{u=0}^{1}\int_{v=0}^{1-u} [\qquad\qquad]\, du\,dv = \frac{1}{12}\left[a^3 b + a^2 bc + b^3 c + ac(b^2 + bc + c^2)\right]$$

$$\int_{\sigma_2^1} \omega = -b^3 c \int_{u=0}^{1}\int_{v=0}^{1-u} u^2\, du\,dv = -\frac{b^3 c}{12}$$

$$\int_{\sigma_2^2} \omega = -ac^3 \int_{u=0}^{1}\int_{v=0}^{1-u} v^2\, du\,dv = -\frac{ac^3}{12}$$

$$\int_{\sigma_2^3} \omega = -a^3 b \int_{u=0}^{1}\int_{v=0}^{1-u} u^2\, du\,dv = -\frac{a^3 b}{12}$$

$$\therefore \int_{C_2} \omega = \frac{abc}{12}(a+b+c)$$

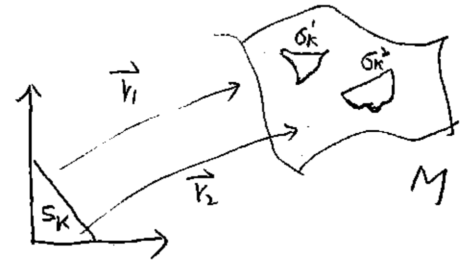
ω : a $\underline{k}$-form on M

$\vec{r}$: $S_{\underline{k}} \to M$

$\vec{r}^{\,*}$: $\Lambda(M) \to \Lambda(S_k)$

$\Lambda(M)$ 是所有 k-form on M

$\vec{r}^{\,*}\omega$: a k-form on S_k



$$\Rightarrow \int_M \omega = \int_{S_K} \vec{r}^{\,*}\omega$$

見 P.12 pull-back 之概念

$\boxed{\text{pull-back}}$

ω : ② -form on R^3 ($k=2$)

$\vec{r}_i$: $R^{②} \to R^3$

例 One can use Gauss div. theorem to calculate the previous example.

$$\int_{C_2 = \partial C_3} \vec{B} \cdot d\vec{S} = \int_{C_3} \vec{\nabla} \cdot \vec{B} \, dV \quad (\text{Gauss theorem}) \quad \text{in } R^3.$$

or in general

$$\boxed{\int_{\partial C_3} \omega = \int_{C_3} d\omega} \quad (\text{Stoke Theorem})$$

for general p-form in R^n

代数拓扑

∂ : boundary operator "∂" on chains ⋯ $\boxed{\text{Homology}}$

d : exterior derivative operator "d" on forms ⋯ $\boxed{\text{Cohomology}}$

- There is a "duality" between "∂" and "d" (1931)

de Rham (1903 ~ 1990 Swiss. under E. Cartan)

(sought-after).

⊛ $\boxed{\text{Homology}}$ Betti number b_k : topological invariants of M.

$\xrightarrow[\text{de Rham}]{}$ Diff forms are related to topology of M !! $\boxed{\text{Cohomology}}$

{ Physics Applications }

$$\sum_{i=0}^{3} \int_{\sigma_2^i} \omega = \int_{\partial c_3} \omega = \int_{c_3} d\omega$$

$$d\omega = 2(x+y+z)\, dx \wedge dy \wedge dz$$

$$\therefore \int_{c_3} d\omega = 2 \int_{x=0}^{a} \int_{y=0}^{-\frac{b}{a}x+b} \int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} (x+y+z)\, dx\,dy\,dz$$

$$= \frac{abc}{12}(a+b+c).$$

迄.

和 P.9 結果相同!

$\boxed{\text{pull-back}}$ Ref: Bachman P.101

i... 演下推廣 P.9 之 pull-back 觀念

例 $\omega = y\,dx + z\,dy + x\,dz$ $\underbrace{\text{1-form on } R^3}_{k=1}$

$\vec{r}(a,b) = (a+b,\ a-b,\ ab)$, $\vec{r}: R^2 \to R^3$

$\vec{r}^*\omega$ is the pull-back of ω under $\vec{r}$: a 1-form on R^2

s.t. $\displaystyle\int_{\vec{r}(\sigma)} \omega = \int_{\sigma} \vec{r}^*\omega$, σ: a 1-chain

$$\Rightarrow \quad \vec{r}^*\omega = (a-b)\,da + (a-b)\,db + ab\,da - ab\,db + (a+b)\,b\,da + (a+b)\,a\,db$$

$$= (a-b+2ab+b^2)\,da + (a-b+a^2)\,db$$

例 $\omega = x^2\,dy\wedge dz + y^2\,dz\wedge dw$ 2-form on R^4

$\vec{r}(a,b,c) = (a,\ b,\ c,\ abc)$, $\vec{r}: R^3 \to R^4$

pull-back

$$\vec{r}^*\omega = a^2\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} db\wedge dc$$

2-form on R^3.

$$+ b^2\begin{vmatrix} 1 & ab \\ 0 & ac \end{vmatrix} dc\wedge db$$

$$+ b^2\begin{vmatrix} 1 & ab \\ 0 & bc \end{vmatrix} dc\wedge da$$

$$= (a^2 - b^2ac)\,db\wedge dc + b^3c\,dc\wedge da$$

- $\boxed{\text{Simplicial complex } K}$ 6.5 Homology group (Algebraic topology. 代数拓樸)
  - $P_0, P_1, P_2, \cdots P_p$ distinct points in R^n
  - $\overline{P_0P_i}$ $(i=1,2,\cdots p)$ be linear independent.
  - A p-simplex $s_p = \left\{ P = \sum_{i=0}^{p} \lambda_i P_i \,,\ \lambda_i \geq 0,\ \sum_{i=0}^{p} \lambda_i = 1 \right\}$
  - Orientations of S_p : 2-classes

$$(P_0 P_1 P_2 \cdots P_p) = s_p \quad \text{even}$$
$$(P_1 P_0 P_2 \cdots P_p) = -s_p \quad \text{odd}$$

Let K be a finite # of simplex in R^2 (can be generalized to R^n)

- building blocks of K are triangles : 3 vertices and 2 edges
- intersection condition :

2 triangles either

① are disjoint or

② have 1 vertex in common or have 2 vertices (and the edge joining them)

例 $K=$  例 $K=$  例 $K=$ 

These are called simplicial complex

例  NOT ok!

- The boundary of the simplex s^p

$$\partial(P_0 P_1 \cdots P_p) = \sum_{i=0}^{p} (-1)^i (P_0 P_1 \cdots \hat{P_i} \cdots P_p)$$

例 $\partial(P_0 P_1) = (P_1) - (P_0)$, $\partial(P_0 P_1 P_2) = (P_1 P_2) - (P_0 P_2) + (P_0 P_1)$

- A p-chain C^p : sum of p-simplices s_i^p

$$C^p = \sum_i a_i s_i^p \quad (a_i \in R) \qquad (a_i \in \mathbb{Z}, \text{ integer p-chain})$$

- $\partial C^p = \sum_i a_i (\partial s_i^p) \leftarrow$ a $(p-1)$-chain.

Theorem : $\partial^2 C^p \equiv \partial(\partial C^p) = 0 \quad (p > 1)$

例 $\partial(\partial(P_0 P_1 P_2)) = \partial(P_1 P_2) - \partial(P_0 P_2) + \partial(P_0 P_1)$

$= (P_2) - (P_1) - ((P_2) - (P_0)) + (P_1) - (P_0) = 0$

- Standard oriented n-simplex in R^n: $\bar{s}^n = (R_0, R_1, \cdots R_n)$

其中 $R_0 = (0, 0, \cdots 0)$, $R_1 = (1, 0, \cdots 0)$, $R_2 = (0, 1, 0, 0 \cdots 0)$, $R_n = (0, 0, \cdots, 1)$

- Triangulation : M : a manifold , K : a simplicial complex

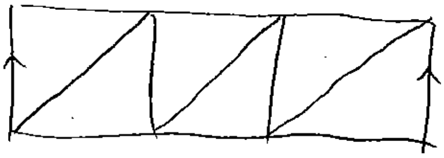
$f: K \to M$ a homeomorphism $\begin{cases} 1-1 \\ 可逆 \end{cases}$

($\boxed{\text{Def}}$ • $f: X_1 \to X_2$ is a homeomorphism if f 连续, f^{-1} 存在且连续

• X_1 is homeomorphic to X_2 若有 $f: X_1 \to X_2$ is a homeomorphism)

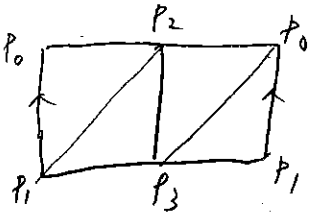
(3)] M : cylinder.

(a)



a triangulation

(b)



$\boxed{\text{Not}}$ a triangulation

$$\sigma_2 = (P_0 P_1 P_2)$$
$$\sigma_2' = (P_2 P_3 P_0)$$
$$\sigma_2 \cap \sigma_2' = (P_0) \cup (P_2)$$

Not a simplex.

- $\boxed{\text{Singular simplex}}$

A (singular) p-chain on M

$$\sigma_p = \sum_i s_i^p$$

Note : ① No geometrical independence.

② No triangulation.

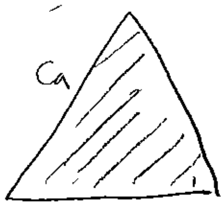
Homology Groups

Motivation:

$M = T^2$

- For the case of simplicial complex K

(a)



C_a

without "hole"

K_1

C_a is a loop, but is a boundary of an area K_1

(b)

a "hole"

K_2 is a loop (without boundary), but NOT boundary of some region.

- $\mathbb{Z}$: a vector space.

  or an abelian group (under +)

  $\mathbb{Z}_2$: cyclic group of order 2

  $$\mathbb{Z}_2 \cong \mathbb{Z}/2\mathbb{Z}$$

  ($N\mathbb{Z}$ is a subgroup of $\mathbb{Z}$ $\Rightarrow$ one can def quotient group $\mathbb{Z}_N \equiv \mathbb{Z}/N\mathbb{Z}$)

Def $G \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \cdots \oplus \mathbb{Z}}_{r} \oplus \mathbb{Z}_{k_1} \oplus \cdots \oplus \mathbb{Z}_{k_p}$

$m = r + p$ generators, r = rank of G.

- <u>Chain group</u>, <u>cycle group</u> and <u>Boundary group</u>.

$\boxed{\text{Chain group } C_r(K)}$: Let there be I_r r-simplexes in K, $\sigma_{r,i}$

(an n-dim simplicial complex)

then $c \in C_r(K)$

$$c = \sum_{i=1}^{I_r} c_i \sigma_{r,i} \qquad c_i \in \mathbb{Z}.$$

group structure : $c + c' = \sum_i (c_i + c_i') \sigma_{r,i}$ (addition of two r-chains)

$$0 = \sum_i 0 \, \sigma_{r,i} \quad \text{(unit)}$$

$$-c = \sum_i (-c_i) \sigma_{r,i} \quad \text{(inverse)}$$

$\Rightarrow$ $C_r(K)$ is an abelian group of rank I_r

$$C_r(K) \cong \underbrace{\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \oplus \mathbb{Z}}_{I_r}, \quad \begin{bmatrix} \mathbb{Z} & & \\ & \mathbb{Z} & \\ & & \mathbb{Z} \end{bmatrix}_{I_r \times I_r}$$

- The <u>Chain complex</u> associated with K : $C(K)$

$$0 \xrightarrow{i} C_n(K) \xrightarrow{\partial_n} C_{n-1}(K) \xrightarrow{\partial_{n-1}} \cdots \xrightarrow{\partial_2} C_1(K) \xrightarrow{\partial_1} C_0(K) \xrightarrow{\partial_0} 0$$

(*) We want to study <u>image</u> and <u>kernel</u> of ∂_r

$r = 0, 1, 2, \cdots, n$.

$\boxed{\text{Cycle group } Z_r(k)}$:

c is a <u>r-cycle</u> if $\partial_r c = 0$, $c \in C_r(k)$.

<u>Def</u> : the set of r-cycles $= Z_r(k)$, a subgroup of $C_r(k)$

Note: $\boxed{Z_r(k) = \ker \partial_r}$ $\quad\quad$ $\left(Z_0(k) = C_0(k) \right)$

$\boxed{\text{Boundary group } B_r(k)}$:

Let $c \in C_r(k)$. if there exists an element $h \in C_{r+1}(k)$

such that $\quad c = \partial_{r+1} h$

then c is called an <u>r-boundary</u>

the set of r-boundaries $= B_r(k)$, a subgroup of $C_r(k)$

Note: $\boxed{B_r(k) = \operatorname{Im} \partial_{r+1}}$ $\quad\quad$ ($B_n(k)$ is df to be 0)

<u>Theorem</u> : $\quad \partial_r(\partial_{r+1} c) = 0$ for any $c \in C_{r+1}(k)$ #

<u>Collary</u> : $\quad B_r(k) \subset Z_r(k)$

- $C_r(K)$, $Z_r(K)$, $B_r(K)$ $\left(K : n\text{-dim simplicial complex}, r \le n \right)$

(*) are they topological invariants or conserved under homeomorphism ??

例

K_1 $\qquad$ K_2

But $\quad C_1(K_1) = \{ i(P_0,P_1) + j(P_1,P_2) + k(P_2,P_0) \mid i,j,k \in \mathbb{Z} \}$

$$\cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z},$$

and $C_1(K_2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

$\therefore$ $\quad C_r(K)$ cannot be a candidate of a topological invariant !

$Z_r(K)$, $B_r(K)$ are NOT either.

Def $\quad \boxed{\text{Homology groups } H_r(K)}$ $\quad$ 又 $\begin{cases} H_r(K, \mathbb{Z}), \\ H_r(K, \mathbb{R}), \\ H_r(K, \mathbb{Z}_2) \ldots \end{cases}$

$$H_r(K) = Z_r(K) / B_r(K), \qquad 0 \le r \le n$$

Theorem : Homology groups $H_r(K)$ are topological invariants.

- One can also def singular homology group of a manifold M

$$\boxed{H_r(M) \equiv Z_r(M) / B_r(M)}$$

with some mild topological assumptions,

singular homology group $\underset{\text{isomorphic to}}{\sim}$ simplicial homology group with R-coefficients.

$\boxed{例}$ 很多，见 Nakahara.

- $H_r(M, \underline{\underline{R}})$

例: $H_1(RP^2) = Z_2$
$H_1(\text{klein bottle}) = Z \oplus Z_2$

$$H(M, Z) \cong Z \oplus Z \cdots \oplus Z \oplus \underbrace{Z_{k_1} \oplus \cdots \oplus Z_{k_p}}_{\text{torsion subgroup}}$$

(描述 twisting)

$$H(M, R) \cong R \oplus R \cdots \oplus R.$$

- $\boxed{\text{Def}}$ $\boxed{b_r(M) \equiv \dim H_r(M, R) = r\text{th Betti number}}$

Betti #s are topological invariants!!

- $\boxed{\text{Def}}$ $\boxed{\text{Euler Characteristic}}$

$$\underset{\substack{\uparrow \\ \text{topological invariant}}}{\chi(M)} \equiv \sum_{r=0}^{n} (-)^r \overset{\substack{\text{\# of } r\text{-simplex} \\ \downarrow}}{I_r} = \sum_{r=0}^{n} (-)^r \overset{\substack{\text{topological invariants} \\ \swarrow}}{b_r(M)}$$

de Rham cohomology groups.

all r-forms

M : a n-dim diff manifold, $\omega \in \Lambda^r(M)$

<u>Def</u> : ω is a <u>closed r-form</u> if $d\omega = 0$ on M.

the set of all closed r-forms on M $= Z^r(M)$ = cocycle group.

<u>Def</u> : ω is an <u>exact r-form</u> if $\omega = d\psi$ for some $\psi \in \Lambda^{r-1}(M)$.

the set of all exact r-form on M

$= B^r(M)$ = coboundary group

<u>Theorem</u> : $\because d^2 = 0 \Rightarrow Z^r(M) \supset B^r(M)$

<u>Def</u> : $$\boxed{H^r(M, R) \equiv Z^r(M) / B^r(M)} \qquad r = 0, 1, 2, \cdots n.$$

de Rham cohomology groups.

- <u>de Rham's Theorem</u>

One can def an inner product between $H_r(M)$ and $H^r(M)$

$$H_r(M) \times H^r(M) \to R$$

$$\Lambda : ([c], [w]) \equiv (c, w) = \int_c w .$$

derivative operator is the adjoint of the boundary operator ∂

$$(c, dw) = (\partial c, w) \quad (\text{Stokes' Theorem})$$

s.t. Λ is bilinear & non-degenerate.

$\therefore$ $H^r(M)$ is the dual vector space of $H_r(M)$

and $\boxed{\dim H^r(M) = \dim H_r(M)}$ #

$$\Longrightarrow \quad \chi(M) = \sum_{r=1}^{n} (-)^r b^r(M)$$

topology $\qquad$ analysis !!

$\boxed{\text{例}}$ $M = \mathbb{R}^2 - \{0\}$

$b_1 = 1 = b^1$

The only 1-form on M $\in H^1(M)$

$$\tau_1 = \frac{-y\,dx + x\,dy}{x^2+y^2}$$

#

$\boxed{\text{例}}$ $M = T^2$

$b_1 = 2 = b^1$

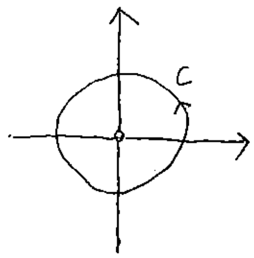
$\tau_1 = d\theta$, $\tau_2 = d\varphi$ are $\overbrace{2 \text{ independent}}^{\text{the only}}$

1-form on M $\in H^1(M)$

#

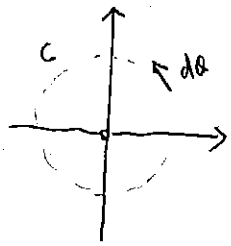
$\boxed{\text{Note}}$

$M = \mathbb{R}^2 - \{0\}$.



$b_1 = 1$

$b^1 = 1$



$\omega = r\,d\theta =$ "$d\theta$" (選 $r \equiv 1$)

"$d\theta$" = "$d \tan^{-1}\frac{y}{x}$" $= \frac{-y\,dx + x\,dy}{x^2+y^2}$

(8) $\boxed{\text{Dirac Monopole}}$

Ref: D.Martin P.306
Nakahara P.15, 131, 137, 266, $\boxed{322}$

Dirac (1931) 考慮磁單極的存在.

$$\nabla\cdot\vec{B} = 4\pi g \quad , \quad \vec{B} = -\nabla \frac{g}{r}$$

① 證明 $\vec{B}$ 不再能寫成

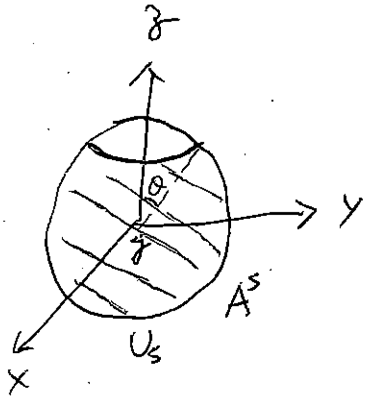
$$\vec{B} = \nabla\times\vec{A}$$

(Hint: $4\pi g = \oint \vec{B}\cdot d\vec{S} = \oint_S \nabla\times\vec{A}\cdot d\vec{S} \overset{\text{由於 stoke theorem}}{=} \int_C \vec{A}\cdot d\vec{l} \to 0$ as $C \to$ a point

see P.8 (r, θ, φ)

(稱 patch).)

② 若僅考慮如圖斜線區域的球面 U_s



計算 $\displaystyle\int_S \vec{B}\cdot d\vec{S} = \frac{g}{r^2}\, 2\pi r^2 (1+\cos\theta)$

$$\therefore \quad A^S_\varphi = -\frac{g(1+\cos\theta)}{r\sin\theta}$$

③ 由 ② 知

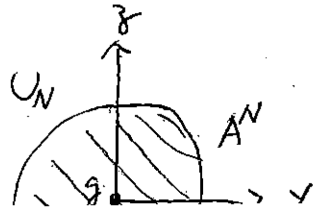
$$\vec{B} = \nabla\times\vec{A} + 4\pi g\, \delta(x)\,\delta(y)\,\theta(z)\,\hat{z}$$

$$\theta(z) = \begin{cases} 1 & z>0 \\ 0 & z<0 \end{cases}$$

上式第二項稱為 Dirac string.

④ 事實上 Dirac string 可由 fibre bundle (纖維叢) 想法去除.

如圖計算 $\displaystyle A^N_\varphi = \frac{g(1-\cos\theta)}{r\sin\theta}$



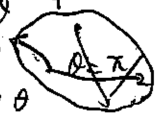
⑤ 証明 total flux

$$\Phi = \oint_S \vec{\nabla}\times\vec{A}\cdot d\vec{S} = \int_{U_N} + \int_{U_S} = 4\pi g .$$

$$= \oint_{\text{equator}} - \oint_{\text{equator}}$$

⑥ 在赤道上

$$\vec{A}^N - \vec{A}^S = \vec{\nabla}\, 2g\varphi = \frac{2g}{r\sin\theta}\,\hat{e}_\varphi$$



any θ ($\theta \neq 0, \pi$)

感謝習題(2)中的 U(1) gauge symmetry

$$\begin{cases} \psi \to \psi' = \exp\left(i\frac{e}{\hbar c}\Lambda\right)\psi \\ \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda . \end{cases}$$

$$\Rightarrow \quad \Lambda \equiv 2g\varphi .$$

⑦ 电子的波函数必須是單值

$$\Rightarrow \quad \boxed{\frac{2ge}{\hbar c} = n = \text{整数}} .$$

著名的 Dirac quantization condition

Dirac 發現：假定磁單極的存在.

(在量子力學裏)我們可以解释為什麼电荷(磁荷)是整数个存在的!!!

(9). 在 $M \equiv R^3 - \{(0,0,c) \mid c \in R\}$ 上定义一向量場 (or 1-form).

$\vec{A}(x,y,z) = \left(\frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}, 0\right)$ or $\tau = \frac{-y\,dx + x\,dy}{x^2+y^2}$ = "$d\theta$" = "$d \tan^{-1}\frac{y}{x}$".

(Note: $\tan^{-1}\frac{y}{x}$ is NOT well defined globally on M.)

① 證明 $\nabla \times \vec{A} = 0$ on M or $d\tau = 0$ on M.

② 但 $\vec{A}$ 不能 globaly 寫成一 gradient

$\vec{A} \neq \vec{\nabla} f$. or $\tau \neq df$ globally on M.

why? (Hint: $2\pi = \oint_\gamma \vec{A}\cdot d\vec{l} \neq \oint_\gamma \vec{\nabla} f \cdot d\vec{l} = 0$.) see P.4.

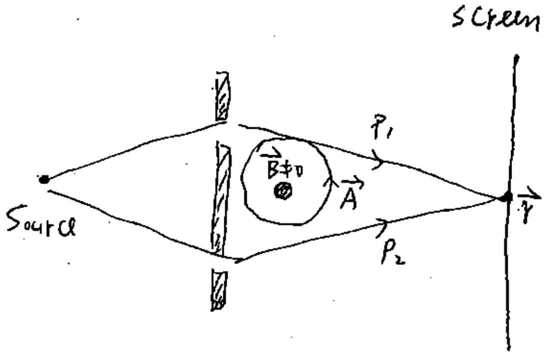
由廣义 stoke theorem $= f(b) - f(a)\big|_{a\to b}$

$\gamma \equiv \{(x,y,0) \mid x^2+y^2=1\}$

- Cohomology.
- The 1st Betti # of M is $b_1 = 1$.

③ 考慮以上在 AB effect 之應用

$$\begin{cases} \vec{A}(\vec{r}) = \left(-\frac{y\phi}{2\pi r^2}, \frac{x\phi}{2\pi r^2}, 0\right), \\ A_0(\vec{r}) = 0. \end{cases}$$



證明 $\begin{cases} \vec{B} = \vec{\nabla}\times\vec{A} = 0 \text{ for } \vec{r} \neq 0, \\ \int \vec{B}\cdot d\vec{s} = \int (\nabla\times\vec{A})\cdot d\vec{s} = \phi\ (flux) = \oint_{P_2 - P_1} \vec{A}\cdot d\vec{l}. \end{cases}$

④ $W \equiv e^{iq\oint \vec{A}\cdot d\vec{l}} = e^{iq\oint A_\mu dx^\mu}$ (Wilson loop)

is a gauge-invariant $\boxed{\text{Observable}}$!!

(8) 在 Minkowski 空間 $R^{1,3}$, $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$, $\varepsilon^{0123} \equiv 1$

計算
$$\left\{ \begin{array}{l} **1 \\ **dx \\ **dx\wedge dy \\ **dx\wedge dy\wedge dz \\ **dt\wedge dx\wedge dy\wedge dz \end{array} \right. = ?$$

(9) 將 Maxwell eq 寫成 diff form 的形式
及电荷守恒 $R^{1,3}$, $\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix}$。

(10) 有一定义在 $D \equiv R^3 - \{(0,0,0)\}$ 的向量場

$$\vec{B} = \frac{1}{r^3}(x, y, z), \quad r \neq 0.$$

ⓐ 証明 $\nabla\cdot\vec{B} = 0$ on D.

ⓑ 是否存在 $\vec{A}$, 使得 $\vec{B} = \nabla\times\vec{A}$ on D ?
若有, 計算 $\vec{A}$, 若無, 証明之。

ⓒ 將 ⓐⓑ 以 diff form 的形式表達之.

Hint: 定义 2-form $\tau = (x\,dy\wedge dz + y\,dz\wedge dx + z\,dx\wedge dy)\frac{1}{r^3}$