physics of Fibre bundles

E&M theory : a vector field on the base manifold $M \equiv R^2$

$$\vec{E}(\underbrace{x,y}_{p}) = (xy^2, xy^3)$$

In Math : The collection of all tangent spaces of $M = R^2$

$$TM = \bigcup_{p\in M} T_pM \qquad \underset{=TM}{\overset{(\text{total space})}{E}} \xrightarrow[\text{projection}]{\pi} M$$

A section (or a vector field) of TM is a smooth map

$$s : M \to TM \quad \text{such that} \quad \pi s = Id_M \quad (\text{or } s(p) = \pi^{-1}(p) = u \in TM)$$

$$p \to \pi^{-1}(p)$$

For this case : $TM = \underset{M}{R^2} \times \underset{F}{R^2}$

a trivial bundle is a direct product

$$\underset{\text{base}}{M} \times \underset{\text{fibre}}{F}$$

• A simple non-trivial fibre bundle

$M = S^1$, $F = [-1, 1]$ = a line segment

$E \xrightarrow{\pi} S^1$ structure group $= G = Z_2 = \{e, g\}$

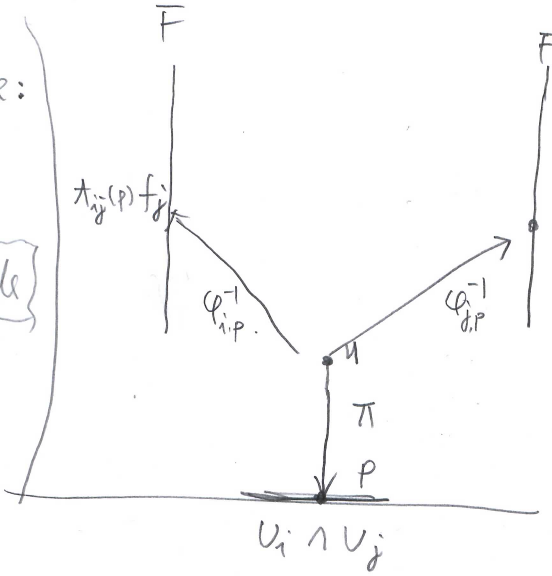
For the important case:

$F \equiv G$

$\Rightarrow$ $\boxed{\text{principal fibre bundle}}$

$\Rightarrow$ connections , physics.

....



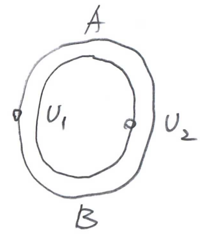
t_{ij} : transition function

φ_i : local trivialization

$\{U_i\}$: an open covering of M.

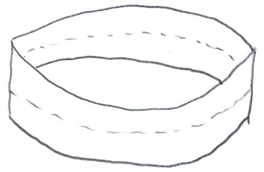
Let $U_1 = (0, 2\pi)$, $U_2 = (-\pi, \pi)$ $\leftarrow$ a covering of S^1

and $A = (0, \pi)$, $B = (\pi, 2\pi)$, the intersection $U_1 \cap U_2$.

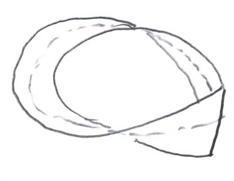


(I) $\varphi_1^{-1}(u) = (\theta, t)$, $\theta \in A$; $\varphi_2^{-1}(u) = (\theta, t)$, $\theta \in B$

(II) $\varphi_1^{-1}(u) = (\theta, t)$, $\theta \in A$; $\varphi_2^{-1}(u) = (\theta, -t)$, $\theta \in B$

(I) 

cylinder, trivial

(II) 

Möbius strip

Complex Manifolds

Recall: One can easily def a vector field on the plane R^2.

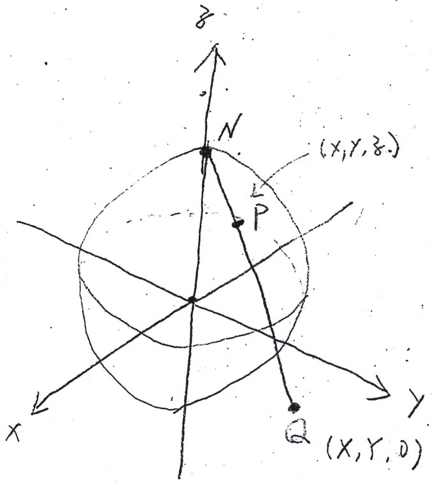
例 $\vec{E}(x,y) = (xy^2, xy^2)$ where (x,y): coordinate of R^2.

But how can one def a vector field on S^2?

Martin p.311

⊛ { A cross-section of the tangent bundle TS^2 }

It turns out that one needs to introduce at least two coordinate systems, say 2 stereographic coordinates on S^2.



- Stereographic coordinate (X,Y) of S^2

$x = \sin\theta\cos\varphi$, $y = \sin\theta\sin\varphi$, $z = \cos\theta$

$$\begin{cases} \theta = \tan^{-1}\left(\dfrac{(x^2+y^2)^{1/2}}{z}\right), & \varphi = \tan^{-1}\dfrac{y}{x} \\ X = \dfrac{x}{1-z}, & Y = \dfrac{y}{1-z} \end{cases}$$

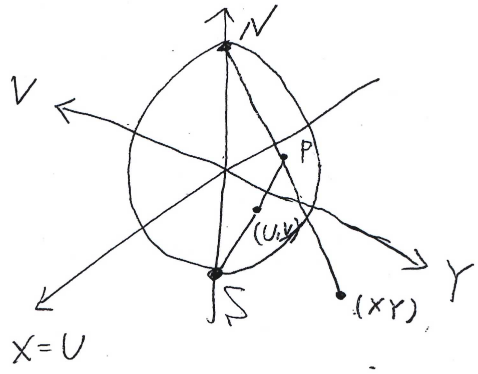
$$\Rightarrow \quad X = \cot\frac{\theta}{2}\cos\varphi, \quad Y = \cot\frac{\theta}{2}\sin\varphi$$

Note: (θ, φ) 不好座標: ① φ: $0 \to 2\pi$ 不連續

(X,Y) 也不好 ： N 點座標 = ??

One way out！ 引進 2 組座標 (X,Y) & (U,V)

(X,Y) 可以涵蓋 $S^2-\{N\}$
(U,V) …… $S^2-\{S\}$



那麼 (X,Y) 跟 (U,V) 的關係是？

P 點可用 (X,Y) or (U,V) 座標表之。

$\boxed{\text{Ans}}$
$$\begin{cases} (X,Y) = \left(\dfrac{x}{1-z}, \dfrac{y}{1-z}\right) \\ (U,V) = \left(\dfrac{x}{1+z}, \dfrac{-y}{1+z}\right) \end{cases}$$
: Cover S^2 with an atlas of two charts

trick： 令
$$\begin{cases} Z = X+iY, \ \bar{Z} = X-iY \\ W = U+iV, \ \bar{W} = U-iV \end{cases}$$

則
$$W = \frac{x-iy}{1+z} = \frac{1-z}{1+z}(X-iY) = \frac{X-iY}{X^2+Y^2} = \frac{1}{Z}$$

$$\therefore \quad U(X,Y) = \frac{X}{X^2+Y^2}$$
$$V(X,Y) = \frac{-Y}{X^2+Y^2}$$

$U(X,Y)$ 與 $V(X,Y)$ 在赤道附近均為 C^∞ 函數

smoothly！

Now we can def a vector field on S^2

by using two charts (X, Y) and (U, V)

$\boxed{例}$ 驗証 $X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}$, $-U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V}$

定义了一個 vector field on S^2

proof : We need to prove that the two expressions agree on the intersection of the two coordinate systems.

$$-U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V} = -\frac{X}{X^2+Y^2}\left(\frac{\partial X}{\partial U}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial U}\frac{\partial}{\partial Y}\right)$$
$$-\left(\frac{-Y}{X^2+Y^2}\right)\left(\frac{\partial X}{\partial V}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial V}\frac{\partial}{\partial Y}\right).$$

$\boxed{\frac{\partial}{\partial X}}$: $$-\frac{X}{X^2+Y^2}\frac{V^2-U^2}{[U^2+V^2]^2} + \frac{Y}{X^2+Y^2}\frac{-2UV}{[U^2+V^2]^2}$$

$$= -X(X^2+Y^2)(V^2-U^2) - Y(X^2+Y^2)2UV$$

$$= X^2+Y^2\left[-X\frac{Y^2-X^2}{[X^2+Y^2]^2} + Y\frac{2XY}{[X^2+Y^2]^2}\right]$$

$$= \frac{XY^2+X^3}{X^2+Y^2} = X.$$

$$\boxed{\frac{\partial}{\partial Y}} : \quad -\frac{X}{X^2+Y^2}\,\frac{2UV}{[U^2+V^2]^2} + \frac{Y}{X^2+Y^2}\,\frac{V^2-U^2}{[U^2+V^2]^2}$$

$$= -X(X^2+Y^2)\,2UV + Y(X^2+Y^2)(V^2-U^2)$$

$$= X^2+Y^2\left[-X\,\frac{(-2XY)}{[X^2+Y^2]^2} + Y\cdot\frac{Y^2-X^2}{[X^2+Y^2]^2}\right]$$

$$= \frac{YX^2+Y^3}{X^2+Y^2} = Y$$

∴ 在赤道附近

$$-U\frac{\partial}{\partial U} - V\frac{\partial}{\partial V} = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y} \qquad \#$$

Note

There are 2 zeros of the vector field.

North pole $(U, V) = (0, 0)$

south pole $(X, Y) = (0, 0)$ ("hairy ball" theorem)

- A vector bundle TS^2: the tangent bundle of S^2

an open covering of S^2: $U_N = S^2 - \{S\}$, $U_S = S^2 - \{N\}$

(X, Y) be the stereographic coordinats of U_N

(U, V) - - - - - - - - - - U_S

$$\Rightarrow \quad U = \frac{X}{X^2+Y^2}, \quad V = \frac{-Y}{X^2+Y^2}$$

$u \in TM$, $\pi(u) = p \in U_N \cap U_S$,

$\varphi_N^{-1}(u) = (p, V_N^\mu)$, $\varphi_S^{-1}(u) = (p, V_S^\mu)$ ← local trivializations

the transition function is

$$t_{NS}(p)^{-1} = t_{SN}(p) = \frac{\partial(U,V)}{\partial(X,Y)} = \begin{pmatrix} \frac{\partial U}{\partial X} & \frac{\partial U}{\partial Y} \\ \frac{\partial V}{\partial X} & \frac{\partial V}{\partial Y} \end{pmatrix}$$

$\begin{pmatrix} X = r\cos\theta \\ Y = r\sin\theta \end{pmatrix}$

$$= r^{-2} \begin{pmatrix} -\cos 2\theta & -\sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{pmatrix}.$$

$= 4\pi$ for $\theta = 2\pi$

2θ !!!

- $\frac{\partial U}{\partial X} = \frac{\partial r}{\partial X}\frac{\partial U}{\partial r} + \frac{\partial \theta}{\partial X}\frac{\partial U}{\partial \theta} = \cos\theta\left(-\frac{\cos\theta}{r^2}\right) + \left(-\frac{\sin\theta}{r}\right)\left(\frac{-\sin\theta}{r}\right) = \frac{-\cos 2\theta}{r^2}$
- $\frac{\partial U}{\partial Y} = \frac{\partial r}{\partial Y}\frac{\partial U}{\partial r} + \frac{\partial \theta}{\partial Y}\frac{\partial U}{\partial \theta} = \sin\theta\left(-\frac{\cos\theta}{r^2}\right) + \left(\frac{\cos\theta}{r}\right)\left(\frac{-\sin\theta}{r}\right) = \frac{-\sin 2\theta}{r^2}$
- $\frac{\partial V}{\partial X} = \frac{\partial r}{\partial X}\frac{\partial V}{\partial r} + \frac{\partial \theta}{\partial X}\frac{\partial V}{\partial \theta} = \cos\theta\left(\frac{\sin\theta}{r^2}\right) + \left(-\frac{\sin\theta}{r}\right)\left(\frac{-\cos\theta}{r}\right) = \frac{\sin 2\theta}{r^2}$
- $\frac{\partial V}{\partial Y} = \frac{\partial r}{\partial Y}\frac{\partial V}{\partial r} + \frac{\partial \theta}{\partial Y}\frac{\partial V}{\partial \theta} = \sin\theta\left(\frac{\sin\theta}{r^2}\right) + \left(\frac{\cos\theta}{r}\right)\left(\frac{-\cos\theta}{r}\right) = \frac{-\cos 2\theta}{r^2}$

- The Hopf fibration (一) 知道 S^3

S^3 is an $U(1) \equiv S^1$ bundle over S^2. (1931)

- 先 change notation of paramitrization of S^2

$$(x, y, z) \to (\xi^1, \xi^2, \xi^3), \quad (\xi^1)^2 + (\xi^2)^2 + (\xi^3)^2 = 1$$

- Need parametrization of S^3:

$$(x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2 = 1$$

Def : $Z^0 \equiv x^1 + i x^2$, $Z^1 \equiv x^3 + i x^4 \Rightarrow |Z^0|^2 + |Z^1|^2 = 1$

✡ If we do the following identification

$$\underbrace{Z \equiv \frac{\xi^1 + i\xi^2}{1 - \xi^3}}_{S^2} = \overbrace{\frac{x^1 + i x^2}{x^3 + i x^4}}^{S^3} \equiv \frac{Z^0}{Z^1}, \quad \text{for } \xi \in U_S,$$

$$W \equiv \overbrace{\frac{\xi^1 - i\xi^2}{1 - \xi^3}}^{S^2} = \overbrace{\frac{x^3 + i x^4}{x^1 + i x^2}}^{S^3} \equiv \frac{Z^1}{Z^0} \quad \text{for } \xi \in U_N,$$

we get the Hopf map $\pi : S^3 \to S^2$

$$\begin{cases} \xi^1 = 2(x^1 x^3 + x^2 x^4) \\ \xi^2 = 2(x^2 x^3 - x^1 x^4) \\ \xi^3 = (x^1)^2 + (x^2)^2 - (x^3)^2 - (x^4)^2 \end{cases}$$

- $\boxed{\text{Observation}}$

Z is invariant under $(z^0, z^1) \to (e^{i\theta} z^0, e^{i\theta} z^1)$

where $e^{i\theta} \in U(1)$ $\Rightarrow$ $(e^{i\theta} z^0, e^{i\theta} z^1) \in S^3$

$$\therefore \quad \begin{matrix} (z^0 \ z^1) \\ (e^{i\theta} z^0, e^{i\theta} z^1) \end{matrix} \overset{\pi}{\rightrightarrows} (\xi^1, \xi^2, \xi^3)$$

Indeed for all $(X^1 X^2 X^3 X^4)$ on S^3 with

$$\frac{X^1 + iX^2}{X^3 + iX^4} = \frac{x^1 + ix^2}{x^3 + ix^4} \quad (x^1, \cdots x^4 \text{ fixed})$$

or
$$\begin{cases} x^3 X^1 - x^4 X^2 - x^1 X^3 + x^2 X^4 = 0 \\ x^4 X^1 + x^3 X^2 - x^2 X^3 - x^1 X^4 = 0 \end{cases} \quad \cdots\cdots (*) ,$$

the images under π are the same $(\xi^1 \xi^2 \xi^3)$.

But $(*)$ is a $\underbrace{\text{great circle}}_{\substack{\| \\ F \leftarrow \text{fibre} \\ \| \\ S^1}}$ of S^3.

$\boxed{\text{推廣}}$

- $\pi : S^7 \to S^4$ quaternions , $F = SU(2) = S^3$
- $\pi : S^{15} \to S^8$ Octonions (Cayley numbers) , $F = S^7$, but NOT a group!

$$\boxed{\begin{matrix} \pi : S^{2n-1} \to S^n \\ n = 2, 4, 8 \end{matrix}}$$

- The fibre bundle structure　　(二) 不知道 S^3

local trivialisations:

$$\varphi_S^{-1} : \pi^{-1}(U_S) \to U_S \times U(1)$$
$$(z^0, z^1) \to \left(z^0/z^1 , \; z^1/|z^1| \right),$$

$$\varphi_N^{-1} : \pi^{-1}(U_N) \to U_N \times U(1)$$
$$(z^0, z^1) \to \left(z^1/z^0 , \; z^0/|z^0| \right).$$

On the equator $\xi^3 = 0 \Rightarrow |z^0| = |z^1| = \frac{1}{\sqrt{2}}$

$\therefore$ on the equator $\varphi_S^{-1} : (z^0, z^1) \to (z^0/z^1 , \sqrt{2} z^1)$,
$\varphi_N^{-1} : (z^0, z^1) \to (z^1/z^0 , \sqrt{2} z^0)$.

The transition function on the equator is

$$t_{NS}(\xi) = \frac{\sqrt{2}\, z^0}{\sqrt{2}\, z^1} = \xi^1 + i \xi^2 \in U(1)$$

$\therefore$ Topological Charge $Q = 1$ of $\pi_1(U(1)) = \mathbb{Z}$ group.

$S^1 \to S^1$

- Alternative description of Hopf map

(三) 知道 S^3 $\Rightarrow$ 推廣 S^7

$S^3 \longleftrightarrow$ a complex 1-sphere S^1_c

$$S^1_c = \{ (z^0, z^1) \in \mathbb{C}^2 \mid |z^0|^2 + |z^1|^2 = 1 \}$$

Def $\pi : S^1_c \to CP^1$

$$(z^0, z^1) \to \{[z^0, z^1]\} = \{ \lambda(z^0, z^1) \mid \lambda \in \mathbb{C}, \lambda \neq 0 \}$$

points of S^3 of the form $\lambda(z^0, z^1)$, $|\lambda| = 1$ are mapped to a single point of $CP^1 = S^2$ 。

- Dirac monopole (1931 !! same year with Hopf)

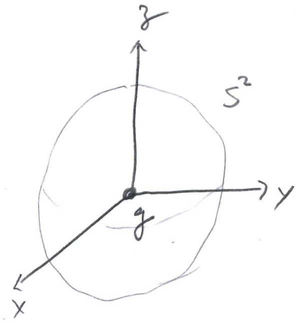
(四) 解物理 $\vec{A}$

physical realization of Hopf!

- Maxwell theory is a U(1) Gauge Theory.

- To avoid Dirac string,

one needs to use 2-patches to describe monopole fields:



g: magnetic monople.

1. $A^N_x = \dfrac{-gy}{r(r+z)}$, $A^N_y = \dfrac{gx}{r(r+z)}$, $A^N_z = 0$.

$\Rightarrow$ one has $\nabla \times \vec{A} = \vec{B}$ except along the negative-Z axis.

2. $A^S_x = \dfrac{gy}{r(r-z)}$, $A^S_y = \dfrac{-gx}{r(r-z)}$, $A^S_z = 0$

$\Rightarrow$ one has $\nabla \times \vec{A} = \vec{B}$ except along the positive-Z axis.

(HW) In polar coordinates
$$\begin{cases} \vec{A}^N(\vec{r}) = \dfrac{g(1-\cos\theta)}{r\sin\theta}\,\hat{e}_\varphi \\ \vec{A}^S(\vec{r}) = -\dfrac{g(1+\cos\theta)}{r\sin\theta}\,\hat{e}_\varphi \end{cases}, \quad \hat{e}_\varphi = -\sin\varphi\,\hat{e}_x + \cos\theta\,\hat{e}_y.$$

$$\vec{A}^N - \vec{A}^S = \vec{\nabla}\, \overbrace{2g\phi}^{\equiv \Lambda}. \quad (\text{on equator})$$

G.T. $\left\{ \psi^S(\vec{r}) = e^{\frac{-ie\Lambda}{\hbar c}} \psi^N(\vec{r}) \right.$ ⟸ electron wave function (a section)

wave function single-valued

$$\Rightarrow \quad \frac{e}{\hbar c}(2g \cdot \underbrace{2\pi}_{\Delta\phi}) = 2\pi n.$$

$$\Rightarrow \quad \boxed{\frac{2eg}{\hbar c} = n} \leftarrow \text{整数}.$$

Dirac quantization condition $\Rightarrow$ ⊛ e quantized !!!

$n \equiv 1$ for magnetic monopole.

Monopole bundle $\Leftrightarrow$ $\underset{= S^1}{U(1)}$ principal bundle over S^2.

Dirac (1931) Hopf (1931)

Wu-Yang (1968)

Hopf map $S^7 \to S^4$

(三) 知道 S^7.

quaternion

$i^2 = j^2 = k^2 = -1$

$ij = -ji = k$
$jk = -kj = i$
$ki = -ik = j$

$$\begin{cases} q = t + xi + yj + zk \\ |q| = (t^2 + x^2 + y^2 + z^2) \end{cases}$$

$|q| = 1$ 表 $SU(2) = S^3$

S^7 is a quaternion one-sphere S^1_H

$$S^1_H = \{ (q^0, q^1) \in H^2 \mid |q^0|^2 + |q^1|^2 = 1 \}$$

Def $\pi : S^1_H \longrightarrow HP^1$

$$(q^0, q^1) \to \{[q^0, q^1]\} = \{ \eta (q^0, q^1) \mid \eta \in H, \eta \neq 0 \}.$$

The transition function defined by this Hopf map belongs to the class 1 of $\pi_3(SU(2)) = \mathbb{Z}$ ← 1-instanton

points of S^7 of the form $\eta(q^0, q^1)$, $|\eta| = 1$ are mapped to a single point of $HP^1 \equiv S^4$.

- $\underline{\text{The fibre bundle structure of instanton}}$ (=) 不知道 S7

$$\begin{cases} U_N = \{ (x, y, z, t) \mid x^2+y^2+z^2+t^2 \leq R^2 + \varepsilon \} \\ U_S = \{ (x, y, z, t) \mid x^2+y^2+z^2+t^2 \geq R^2 - \varepsilon \} \end{cases}$$

at a point $P = (x, y, z, t) \in U_N \cap U_S$, $\pi^{-1}(P) = u$.

The $\underline{\text{local trivializations}}$

$$\varphi_N^{-1}(u) = (P, g_N), \quad \varphi_S^{-1}(u) = (P, g_S)$$

where $g_N, g_S \in SU(2)$

on $U_N \cap U_S$, we have $g_N = \frac{1}{R}(tI + i x^i \sigma_i)\, g_S$ the large gauge transformation.

where $\dfrac{tI + i x^i \sigma_i}{\sqrt{x^2+y^2+z^2+t^2}\ (=R)}$ is the $\underline{\text{transition function}}$.

(or large gauge transf)

$Q = 1$ of $\pi_3(SU(2)) = \mathbb{Z}$ group.

1-Instanton bundle

(例) 解 for SDYM

't Hooft symbol

$$A_\mu^1(x) = \eta_{\mu\nu} \frac{(x-a)_\nu}{|x-a|^2+\lambda^2}$$

$$F_{\mu\nu}^1(x) = \eta_{\mu\nu} \frac{\lambda^2}{[|x-a|^2+\lambda^2]^2}$$

$$Q = 1$$