LINEAR PHASE COSINE MODULATED MAXIMALLY DECIMATED FILTER BANKS WITH PERFECT RECONSTRUCTION †

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Abstract. In this paper a new type of maximally decimated FIR cosine modulated filter banks is proposed. Each analysis and synthesis filter in this filter bank has linear phase. We can design the system to have approximate reconstruction property (pseudo-QMF system) or perfect reconstruction property (PR system). The filter bank is paraunitary in the PR case.

Although there are $2M$ channels in the new system, the cost (in terms of design and implementation complexity) is comparable to that of an M channel system. Correspondingly, the coding gain of the new system is also comparable to that of a traditional M channel system (rather than a $2M$ channel system). Examples will be given to demonstrate that very good attenuation characteristics can be obtained with the new system.

1. Introduction

In [1], three approaches were suggested for designing cosine modulated filter banks with approximate aliasing cancellation. The first one involves designing two prototype filters. The implementation cost for the analysis bank is that of two prototype filters plus cosine and sine modulation. The second method, similar to the one proposed earlier by Rothweiler [2], requires only one prototype filter. Its distortion $T(z)$ has linear phase and approximately flat magnitude response, but individual analysis and synthesis filters do not have linear phase, which is important for image coding applications. The third method, given in [1], needs also only one prototype filter. With this method all the analysis and synthesis filters have linear phase, but the resulting $|T(e^{j\omega})|$ has a peak or a null at zero frequency and at π.

Recently, some cosine modulated maximally decimated systems with perfect reconstruction property have been proposed [3]-[6]. These have all the advantages of cosine modulated systems and enjoy perfect reconstruction property. But individual analysis and synthesis filters do not have linear phase, even though the prototype filters do. In [7], some techniques were developed to characterize and design paraunitary linear phase filter bank, but these are *not* cosine modulated.

In general, the following results are typically desired in a filter bank:

1. *Aliasing errors*: Exact or approximate alias cancelation.
2. *Distortion function*: $T(z)$ is exactly or nearly a delay; $T(z)$ has approximately flat magnitude response and linear phase.
3. *Cosine modulation*: All filters must be cosine modulated versions of a prototype filter. This helps to get design and implementation efficiency.
4. *Linear phase analysis and synthesis filters*: These are desired in image coding applications, when the subbands are heavily quantized. (The nonlinearity of phase of individual filters leads to some artifacts in the reconstructed image).

In a (nearly or exactly) perfect reconstruction system, properties 2,3 and 4 have not been simultaneously achieved in the past. Reference [2] achieved property 2 and 3. Reference [1] achieved property 3 and 4. In this paper, we will show how to achieve all of these.

2. The new cosine modulated system

The set up of this new maximally decimated $2M$ channel filter bank is shown in Fig. 2.1. It is a connection of two subsystems. The first subsystem has $M+1$ channels and the second subsystem has $M-1$ channels. Every filter in this system has linear phase and is cosine modulated version of one prototype filter.

2.1 Approximate reconstruction system

In this case, the new filter bank can be designed to be almost alias free. Aliasing suppression improves as the stopband attenuation of the prototype increases. Its distortion function, $T(z)$, has approximately flat magnitude response and linear phase.

Suppose the prototype filter $P_0(z)$ is of order N and linear-phase. Let

$$U_k(z) = P_0(zW^k), \tag{2.1}$$

where $W^k = e^{-jk\pi/M}$. Magnitude response of $U_k(z)$ is only a shift of $|P_0(e^{j\omega})|$ by $k\pi/M$. Sketches of magnitude responses of $P_0(z)$ and $U_k(z)$ are shown in Fig. 2.2.

† Work supported in parts by NSF grant MIP 92-15785, Tektronix, Inc., and Rockwell International.

Construct the analysis and synthesis filters in Fig. 2.1 as follows:

$$\begin{aligned} H_k(z) &= a_k(U_k(z) + U_{-k}(z)), & k &= 0,1,\ldots,M, \\ H'_k(z) &= jz^{-M}(U_k(z) - U_{-k}(z)), & k &= 1,2,\ldots,M-1, \\ F_k(z) &= z^{-(N+M)}\widetilde{H}_k(z), & k &= 0,1,\ldots,M, \\ F'_k(z) &= z^{-(N+M)}\widetilde{H}'_k(z), & k &= 1,2,\ldots,M-1, \end{aligned} \tag{2.2}$$

where $j = \sqrt{-1}$ and

$$a_k = \begin{cases} \frac{1}{\sqrt{2}}, & \text{if } m = 0 \text{ or } M, \\ 1, & k = 1,\ldots,M-1. \end{cases}$$

Figs. 2.3(a) and (b) show, respectively, symbolic magnitude responses of analysis filters in the two subsystems. Notice that the second subsystem does not have filters with passbands covering zero frequency or π while the first subsystem does. The synthesis filters are time-reversed versions of analysis filters and therefore have identical magnitude responses.

The time domain equivalent of (2.2) is

$$\begin{aligned} h_k(n) &= \sqrt{2}p_0(n)\cos(\frac{\pi}{M}kn), \quad k = 0 \text{ or } M, \\ h_k(n) &= 2p_0(n)\cos(\frac{\pi}{M}kn), \quad k = 1,\ldots,M-1, \\ h'_k(n+M) &= 2p_0(n)\sin(\frac{\pi}{M}kn), \quad k = 1,\ldots,M-1, \\ f_k(n) &= h_k(N+M-n), \quad k = 0,\ldots,M, \\ f'_k(n) &= h'_k(N+M-n), \quad k = 1,\ldots,M-1, \end{aligned} \tag{2.3}$$

where $p_0(n)$ is the impulse response of the prototype filter. The linear phase property of $P_0(z)$ and the modulation in (2.3) ensure that every individual filter has linear phase.

We can verify that if $P_0(z)$ is close to a Nyquist($2M$) filter and has stopband edge $\omega_s < \pi/M$, then this filter bank has the following four properties [8]. (1) $|T(e^{j\omega})|$ is approximately flat. (2) $T(z)$ has linear phase. (3) The system is nearly alias free. (4) Every analysis and synthesis filter has linear phase.

2.2 Perfect reconstruction system

Based on the set up for approximate reconstruction system, we can further constrain the polyphase components of the prototype filter so that the resulting filter bank is paraunitary and hence PR.

Let $G_n(z)$ be the nth Type 1 polyphase component of $P_0(z)$ [9]. It can be shown that if the analysis and synthesis filter are chosen as in (2.3), then the system in Fig. 2.1 is paraunitary if and only if the following are true [8].

$$\widetilde{G}_0(z)G_0(z) = 1, \quad \text{and} \quad \widetilde{G}_M(z)G_M(z) = 1. \tag{2.4}$$

$$\widetilde{G}_k(z)G_k(z) + \widetilde{G}_{k+M}(z)G_{k+M}(z) = 2, \tag{2.5}$$

$k = 1,2,\ldots,M-1$.

Notice that the constraints imposed on the $P_0(z)$ do not affect the linear phase property of individual filters. Consequently, with prototype filter satisfying (2.4) and (2.5) and filters as in (2.3), the $2M$ channel system in Fig. 2.1 is a cosine modulated linear phase PR filter bank.

3. Comparison of the new system to conventional filter banks

In this section, similarities and differences between the new system and conventional cosine modulated filter banks are elaborated.

3.1 Different spectral support

In a conventional N channel maximally decimated cosine modulated PR or approximately PR system, all filters have the same total bandwidth $2\pi/N$ (including positive and negative frequency) and the same height in passband. Their passbands do not overlap significantly. When the subband signals are decimated by N, there is no severe aliasing in the subbands. Aliasing is only due to nonideal nature of filter cutoff.

The new system, however is different from traditional filter banks in all that mentioned above. As shown in Fig. 2.3(a), $H_k(z)$ and $H'_k(z)$ have the same spectral supports and total bandwidth $2\pi/M$, i.e., two times wider than they are in the traditional case while $H_0(z)$ and $H_M(z)$ have total bandwidth only π/M. Also $H_0(z)$ and $H_M(z)$ have $\sqrt{2}$ times the height of the other filters. Serious aliasing occurs in the subbands, but are cancelled in the synthesis bank.

3.2 Application in subband coding

Although there is severe aliasing in the subbands, it is still possible to exploit the energy distribution of the original input signal $x(n)$ in the usual way and obtain coding gain advantage like conventional filter banks.

To explain, let $x_k(n)$ and $x'_k(n)$ denote the outputs of $H_k(z)$ and $H'_k(z)$ respectively, as in Fig. 3.1(a). With filters as in (2.2), we can show that

$$2z^{-M}X(z)U_k(z) = z^{-M}X_k(z) + jX'_k(z), \tag{3.1}$$

where $U_k(z)$ is as defined in (2.1). Eq. (3.1) means we can interpret $x_k(n)$ and $x'_k(n)$ as the real and imaginary parts of the one-sided signal $X(z)U_k(z)$. A pictorial illustration is given in Fig. 3.1(b). So $x_k(n)$ and $x'_k(n+M)$ together retain the usual meaning of subband signals. (A somewhat related result can be found in [10], which came to our attention recently.) From the fact that $H_k(z)$ and $H'_k(z)$ have the same spectral occupancy, we see that the energies of $x_k(n)$ and $x'_k(n)$ are essentially the same. These in turn are proportional to the energy of the hypothetical complex subband signal $y_k(n)$. The decimation of $x_k(n)$ and $x'_k(n)$ by $2M$ is equivalent to decimating $y_k(n)$ by $2M$. But that does not lead to severe aliasing other than due to usual filter nonidealities since $y_k(n)$ has bandwidth $2\pi/2M$. So even though the decimation of the subband signals in Fig. 2.1 creates severe aliasing, it still makes sense to quantize and encode the decimated signals based on the energy distributions of the undecimated signals. We conclude the new filter bank still finds application in subband coding. (This has been verified by performing image coding with the proposed filter bank.)

Coding gain

In the construction of the new PR filter bank, the system is constrained to be paraunitary. The coding gain and optimal bit allocation for paraunitary systems can be found in [9]. However, because of spectral overlapping between the two subsystems, the $2M$ channel filter bank is found to have coding gain close to an M channel rather than a $2M$ channel paraunitary filter bank. (See [8] for detail.)

3.3 Design cost and implementation complexity

The design of the prototype filter in approximate reconstruction system is exactly the same as that in traditional M channel pseudo QMF cosine modulated filter banks [8], [9]. In the PR case, it is sufficient to minimize the stopband energy of the prototype filter under the two conditions in (2.4) and (2.5). This is similar to the case of traditional PR M channel cosine modulated filter banks [4]. Therefore, the $2M$ channel system and usual (QMF or PR) cosine modulated filter banks have the same design cost.

The implementation complexity of a conventional M channel maximally decimated cosine modulated filter bank is that of the prototype filter plus one DCT matrix working at M-fold decimated rate [9]. In both approximate reconstruction and PR case, the proposed $2M$ system has nearly the same cost [8], i.e., number of computations per input sample is nearly the same.

4. Numerical examples

Example 4.1 Approximately PR system with $M = 8$: The prototype filter has order $N = 88$, stopband attenuation 73 dB and stopband edge $\omega_s = 0.12\pi$. Fig. 4.1(a) (b) show, respectively, magnitude responses of the first set of analysis filters and the second set of analysis filters. In this example we have satisfactory suppression of aliasing error, $\sqrt{\sum_{i=1}^{i=2M-1}|A_i(e^{j\omega})|^2}$ (where $A_i(z)$ denotes the alias transfer function of the ith alias component as defined in [9]). The worst peak aliasing error is only about 0.0013. Also the amplitude distortion function is approximately flat with peak amplitude distortion 0.002.

Example 4.2. PR system with $M = 7$: The prototype $P_0(z)$ in this example is of order 49. It has stopband attenuation 42 dB and stopband edge ω_s 0.165π. Fig. 4.2 (a) (b) show, respectively, magnitude responses of the first set of analysis filters and the second set of analysis filters.

In both of these examples, the analysis filters have linear phase by construction, so we have not shown the phase responses.

Summarizing, we have developed a linear phase cosine modulated maximally decimated system with perfect reconstruction. Design examples have shown that very good attenuation characteristics can be obtained as well.

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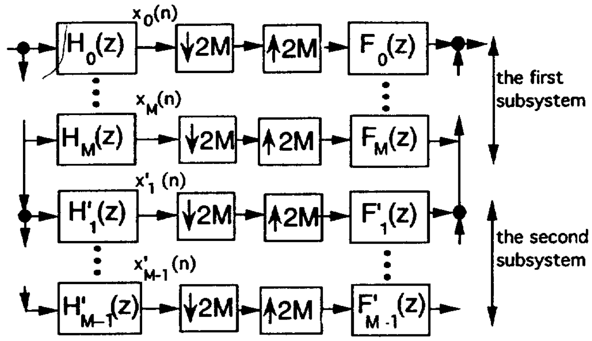

Fig. 2.1 The new setup for derivation of the cosine modulated maximally decimated filter bank.

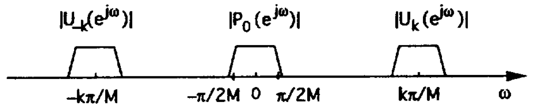

Fig. 2.2 Magnitude responses of $P_0(z)$ and $U_k(z)$.

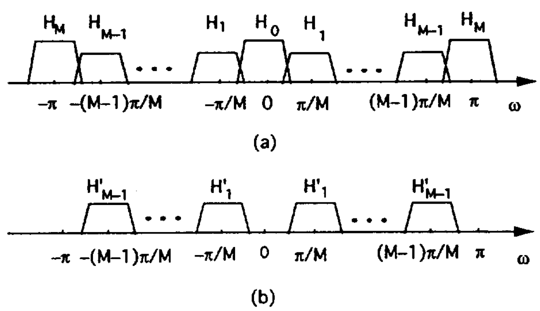

Fig. 2.3 Magnitude responses of the cosine modulated analysis bank filters,(a) the first subsystem, (b) the second subsystem.

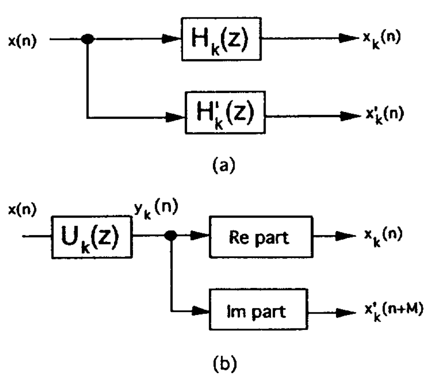

Fig. 3.1 An interpretation of $x_k(n)$ and $x'_k(n+M)$.

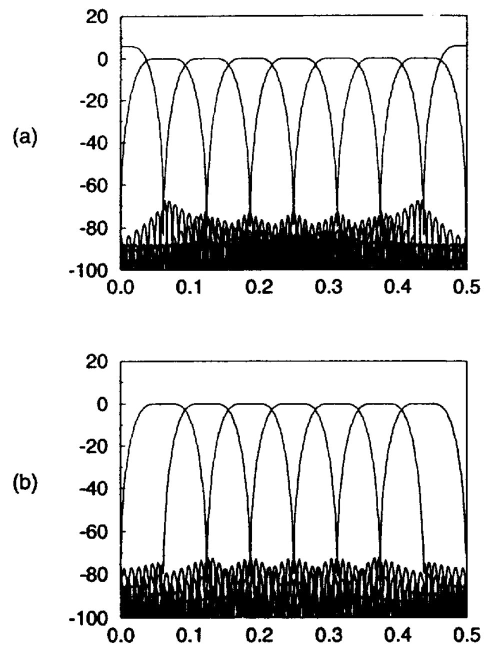

Fig. 4.1. Example 4.1. Approximate PR system with M=8. Magnitude responses of analysis filters in (a) first subsystem, (b) second subsystem.

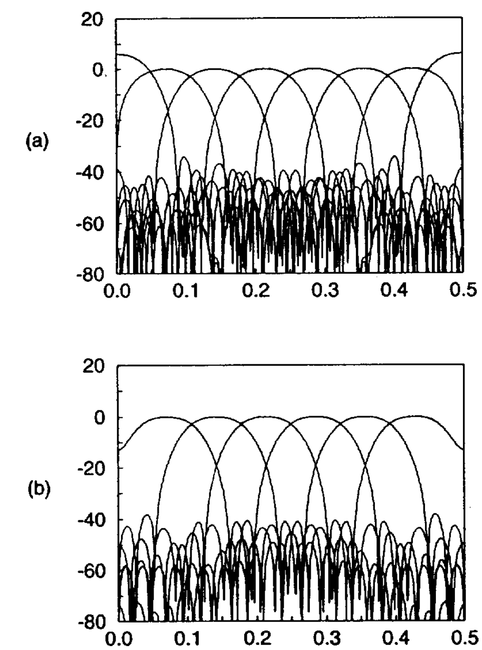

Fig. 4.2. Example 4.2. Cosine modulated PR system with M=7. Magnitude responses of analysis filters in (a) first subsystem, (b) second subsystem.