MINIMAL FACTORIZATION OF LAPPED UNIMODULAR TRANSFORMS

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ABSTRACT

The Lapped Orthogonal Transform (LOT) [1] is a popular transform and has found many applications in signal processing. Its extension, BiOrthogonal Lapped Transform (BOLT), has been investigated in detail in [2]. In this paper, we will study Lapped Unimodular Transform (LUT). All of these three transforms are first-order matrices with FIR inverses. We will show that like LOT and BOLT, all LUTs can be factorized into degree-one unimodular matrices. The factorization is both ***minimal and complete***. We will also show that all first-order systems with FIR inverses can be minimally factorized as a cascade of degree-one LOT, BOLT, and unimodular building blocks. However unlike LOT and BOLT, unimodular filter banks of any order (which include LUTs as a special case) can never have linear phase.

1. INTRODUCTION

Consider the first-order $M \times M$ matrix

$$\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}. \tag{1}$$

Such a matrix is also known as a lapped transform. When the matrix satisfies $\mathbf{A}^T(z^{-1})\mathbf{A}(z) = \mathbf{I}$, it is called LOT [1] [4]. The LOTs have been widely applied in various applications [1]. In this paper, we will study another class of matrices, namely unimodular matrices. A causal FIR matrix $\mathbf{H}(z)$ is unimodular if $[\det \mathbf{H}(z)] = c$ for some $c \neq 0$. Here we will assume that $c = 1$ for notational simplicity. When the unimodular matrix has order one as in (1), it will be called lapped unimodular transforms (LUT). Causal FIR unimodular matrices have the advantage that both their inverses and themselves are causal FIR matrices. If they are used for the polyphase matrices of filter banks (FB), perfect reconstruction (PR) can be obtained by FIR analysis and synthesis filters.

The earliest paper that studied the relationship between unimodular matrices and FIR PR FB is [3]. Using the system-theoretic concepts, the authors showed a number of properties of causal FIR unimodular matrices. In particular, the authors showed that there are examples of second-order unimodular matrices that cannot be factorized into degree-one unimodular matrices. Moreover it was showed that any causal FIR matrix $\mathbf{H}(z)$ with $[\det \mathbf{H}(z)] = cz^{-L}$ can be decomposed into a product of a unimodular matrix and a paraunitary (PU) matrix. Even though such a decomposition is not necessarily minimal, it proved that all FIR PR FBs can be captured by a PU matrix and a unimodular matrix. In [2], the authors introduced the most general degree-one unimodular system. The LUTs were characterized by some matrices with eigenvalues equal to zero. However the factorization of LUTs was not studied.

It is well-known that unimodular matrices can be expressed as a product of elementary matrices (defined and proved in [5]). Elementary matrices can be realized by using the lifting schemes [6] [7]. Lifting schemes enjoy the advantages of having low complexity and being structurally PR. That is, the FB continues to have PR even when the lifting coefficients are quantized. However such a representation is not minimal and not unique.

In this paper, we will show that all LUTs can be factorized into degree-one unimodular matrices. The factorization is both minimal and complete. We will also show that all first order systems with FIR inverses can be minimally factorized as a cascade of degree-one LOT, BOLT, and unimodular building blocks. However unlike LOT and BOLT, unimodular FBs of any order (including the LUT) cannot have linear phase. As a potential application, we will show how unimodular matrices can be used to get a vector DPCM with FIR encoder and FIR decoder.

Notations and Definitions: Boldfaced upper and lower case letters are used to denote matrices and vectors respectively. For a causal polynomial $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1} + \ldots + \mathbf{A}_N z^{-N}$ with $\mathbf{A}_N \neq \mathbf{0}$, its order is equal to N while its degree is the minimum number of delay required to realize the matrix. For example, the matrix $\mathbf{A}(z)$ defined in (1) has order one while its degree is equal to the rank of $\mathbf{A}_1$.

THIS WORK WAS SUPPORTED BY NSC 88-2213-E-002-080 AND 88-2218-E-009-016, TAIWAN, R.O.C.

2. SOME PROPERTIES OF LUT

In this section, we will derive some properties of LUT that will be useful for the derivation of its factorization.

Fact: Let $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1} + \ldots + \mathbf{A}_N z^{-N}$ with $\mathbf{A}_N \neq \mathbf{0}$ be unimodular. Then we have (i) $\mathbf{A}_N$ is singular; (ii) $\mathbf{A}_0$ is nonsingular.
Proof: Suppose $\mathbf{A}^{-1}(z) = \mathbf{B}_0 + \mathbf{B}_1 z^{-1} + \ldots + \mathbf{B}_K z^{-K}$, with $\mathbf{B}_K \neq \mathbf{0}$. By simple multiplication, we have $\mathbf{A}_N\mathbf{B}_K = \mathbf{0}$ and $\mathbf{A}_0\mathbf{B}_0 = \mathbf{I}$. This proves that $\mathbf{A}_N$ is singular and $\mathbf{A}_0$ is nonsingular.

When the matrix $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}$ is an LUT, then $\mathbf{A}_0$ is nonsingular while $\mathbf{A}_1$ is singular. We can always rewrite an LUT as

$$\mathbf{A}(z) = \mathbf{A}_0(\mathbf{I} + \mathbf{P}z^{-1}),$$

for some singular matrix $\mathbf{P}$ and some nonsingular matrix $\mathbf{A}_0$. In the rest of the paper, we will focus on $(\mathbf{I} + \mathbf{P}z^{-1})$.
Linear Phase Unimodular Filter Banks? A filter bank has linear phase if and only if its polyphase matrix $\mathbf{E}(z) = \sum_{i=0}^{N} \mathbf{E}_i z^{-i}$ possesses the symmetry property:

$$\mathbf{E}_{N-i} = \mathbf{E}_i\mathbf{D},$$

where $\mathbf{D}$ is the anti-diagonal matrix:

$$\mathbf{D} = \begin{pmatrix} 0 & \ldots & 0 & \pm 1 \\ 0 & \ldots & \pm 1 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ \pm 1 & 0 & \ldots & 0 \end{pmatrix}.$$

These equations imply that the ranks of $\mathbf{E}_N$ and $\mathbf{E}_0$ are the same. This contradicts fact that for an unimodular FB, $\mathbf{E}_0$ is nonsingular and $\mathbf{E}_N$ is singular. Therefore, we conclude that ***unimodular filter banks can never have linear phase.***

Theorem 1 *An $M \times M$ matrix of the form $(\mathbf{I} + \mathbf{P}z^{-1})$ is unimodular if and only if $\mathbf{P}^M = \mathbf{0}$. Moreover if K is the smallest number such that $\mathbf{P}^K = \mathbf{0}$, then its inverse is given by*

$$\mathbf{I} + (-\mathbf{P})z^{-1} + (-\mathbf{P})^2 z^{-2} + \ldots + (-\mathbf{P})^{K-1} z^{-K+1}.$$

Proof: Suppose that $(\mathbf{I} + \mathbf{P}z^{-1})$ is unimodular. Then there exists a causal FIR matrix $\mathbf{B}(z) = \sum_{i=0}^{L} \mathbf{B}_i z^{-i}$ such that

$$(\mathbf{I} + \mathbf{P}z^{-1})(\mathbf{B}_0 + \mathbf{B}_1 z^{-1} + \ldots + \mathbf{B}_L z^{-L}) = \mathbf{I}.$$

After carrying out some algebraic calculation, we get $\mathbf{P}^{L+1} = \mathbf{0}$, and $\mathbf{B}_i = (-\mathbf{P})^i$ for $i = 0, 1, \ldots, L$. On the other hand, if $\mathbf{P}^M \neq \mathbf{0}$, then from matrix theory we know that $\mathbf{P}^i \neq 0$ for all finite i. Therefore if $\mathbf{P}^M \neq \mathbf{0}$, there does not exist any FIR inverse.

One immediate consequence of the theorem is that the inverse of an LUT has an order of at most $(M-1)$. Using the Schur's unitary triangularization theorem [5], we can always write $\mathbf{P} = \mathbf{T}\triangle\mathbf{T}^\dagger$, for some unitary matrix $\mathbf{T}$ and some lower triangular matrix $\triangle$. The diagonal elements $[\triangle]_{ii}$ are the eigenvalues of $\mathbf{P}$. Since $\triangle^M = \mathbf{0}$ if and only if all of these eigenvalues are zero, we have proved:

Corollary 1: An $M \times M$ matrix $(\mathbf{I} + \mathbf{P}z^{-1})$ is unimodular if and only if all eigenvalues of $\mathbf{P}$ are zero.

Remark: In [2], the authors consider first order degree-ρ matrices of the form $\mathbf{I} - \mathbf{U}\mathbf{V}^\dagger + \mathbf{U}\mathbf{V}^\dagger z^{-1}$, where $\mathbf{U}$ and $\mathbf{V}$ are $M \times \rho$ matrices. It is shown that such a matrix is unimodular if and only if $\mathbf{V}^\dagger\mathbf{U}$ has all the eigenvalues equal to zero.

3. FACTORIZATION OF LUT

We will first derive the most general degree-one unimodular matrix and then show that all LUTs can be factorized into these building blocks. From previous section, we know $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}$ can always be written as $\mathbf{A}_0[\mathbf{I} + \mathbf{P}z^{-1}]$. $\mathbf{A}(z)$ has degree one if and only if $\mathbf{P}$ has rank one. Since $\mathbf{P}$ has rank one, $\mathbf{P} = \mathbf{u}\mathbf{v}^\dagger$ for some nonzero vectors $\mathbf{u}$ and $\mathbf{v}$. From Theorem 1, we know that $[\mathbf{I}+\mathbf{P}z^{-1}]$ is unimodular if and only if $\mathbf{P}^M = \mathbf{0}$. Using the fact that $\mathbf{P}^M = (\mathbf{v}^\dagger\mathbf{u})^{M-1} \cdot \mathbf{u}\mathbf{v}^\dagger$, we conclude that $\mathbf{v}^\dagger\mathbf{u} = 0$. Hence the most general degree-one unimodular matrix is a cascade of a nonsingular matrix $\mathbf{A}_0$ and a building block $\mathbf{D}(z)$ of the form:

$$\mathbf{D}(z) = \mathbf{I} + \mathbf{u}\mathbf{v}^\dagger z^{-1}, \quad \mathbf{v}^\dagger\mathbf{u} = 0. \tag{2}$$

Its inverse is given by $\mathbf{D}^{-1}(z) = \mathbf{D}(-z) = \mathbf{I} - \mathbf{u}\mathbf{v}^\dagger z^{-1}$, which is also a degree-one unimodular system. Using $\mathbf{D}(z)$ as a building block, we are now ready to show the factorization of LUTs.

Theorem 2 *The $M \times M$ matrix $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}$ is a degree-ρ LUT if and only if it can be expressed as*

$$\mathbf{A}(z) = \mathbf{A}_0\mathbf{D}_0(z)\mathbf{D}_1(z)\ldots\mathbf{D}_{\rho-1}(z), \tag{3}$$

where $\mathbf{D}_i(z) = \mathbf{I} + \mathbf{u}_i\mathbf{v}_i^\dagger z^{-1}$ and $\mathbf{A}_0$ is nonsingular. The vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are such that (i) both $\mathbf{U} = [\mathbf{u}_0\ \mathbf{u}_1\ \ldots\ \mathbf{u}_{\rho-1}]$ and $\mathbf{V} = [\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{\rho-1}]$ have full rank; (ii) their product satisfies (here '$\times$' denotes the don't-care term):

$$\mathbf{V}^\dagger\mathbf{U} = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ \times & 0 & 0 & \ldots & 0 \\ \times & \times & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \times & \times & \times & \ldots & 0 \end{pmatrix}. \tag{4}$$

Proof: If $\mathbf{A}(z)$ can be expressed as the product in (3) and (4), it is not difficult to verify that it is a degree-ρ LUT. Suppose that $\mathbf{A}(z)$ is a LUT, then it can be rewritten as $\mathbf{A}(z) = \mathbf{A}_0\mathbf{P}(z)$, where $\mathbf{P}(z) = [\mathbf{I} + \mathbf{P}z^{-1}]$. As $\mathbf{A}(z)$ has degree ρ, rank of $\mathbf{P}$ is also ρ. So there exist $M \times \rho$ matrices $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$ such that $\mathbf{P} = \widetilde{\mathbf{U}}\widetilde{\mathbf{V}}^\dagger$. From Corollary 1, we know that $\mathbf{P}$ has all the eigenvalues equal to zero. Therefore the $\rho \times \rho$ matrix $\widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{U}}$ has all the eigenvalues equal to zero (since any nonzero eigenvalue of $\mathbf{AB}$ is also an eigenvalue of $\mathbf{BA}$.) Using Schur's unitary triangularization theorem, we can find a $\rho \times \rho$ unitary matrix $\mathbf{T}$ such that

$$\mathbf{T}^\dagger\widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{U}}\mathbf{T} = \triangle,$$

for some lower triangular matrix $\triangle$ with all the diagonal elements equal to zero. Letting $\mathbf{U} = \widetilde{\mathbf{U}}\mathbf{T}$ and $\mathbf{V} = \widetilde{\mathbf{V}}\mathbf{T}$, one can verify that

$$\begin{aligned}&\mathbf{I} + \mathbf{P}z^{-1}\\ =&\mathbf{I} + \widetilde{\mathbf{U}}\widetilde{\mathbf{V}}^\dagger\\ =&\mathbf{I} + \mathbf{U}\mathbf{V}^\dagger\\ =&\left[\mathbf{I} + \mathbf{u}_0\mathbf{v}_0^\dagger z^{-1}\right]\dots\left[\mathbf{I} + \mathbf{u}_{\rho-1}\mathbf{v}_{\rho-1}^\dagger z^{-1}\right],\end{aligned}$$

where the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are obtained from $\mathbf{U} = \begin{pmatrix}\mathbf{u}_0 & \mathbf{u}_1 & \dots & \mathbf{u}_{\rho-1}\end{pmatrix}$ and $\mathbf{V} = \begin{pmatrix}\mathbf{v}_0 & \mathbf{v}_1 & \dots & \mathbf{v}_{\rho-1}\end{pmatrix}$. Note that the number of degree-one building blocks in the factorization (3) is equal to ρ, the degree of the LUT. Hence such a factorization is minimal.

A note on the degree of cascasde of unimodular systems: It is well known if we cascade two causal PU matrices of degree ρ_1 and ρ_2, the resulting system is a causal PU system with degree $\rho_1 + \rho_2$. The same is true for the class of CAusal Fir matrix with AntiCAusal Fir Inverse (CAFA-CAFI) [2]. For unimodular matrices, this is no longer true. For example, if we cascade two degree-one unimodular system, namely $\mathbf{D}(z)$ as in (2) and $\mathbf{D}(-z)$, the resulting system is the identity matrix which has a degree of zero! Therefore cascading more unimodular systems does not always result in an unimodular system with a higher degree. However in the LUT case, the degree-one system $\mathbf{D}_i(z)$ in (3) cannot cancel each other, since if they do, the cascade system will have a degree smaller than ρ.

Degrees of freedom: Any $M \times M$ degree-ρ LUT system is characterized by (3). The constant matrix $\mathbf{A}_0$ has M^2 elements, and the 2ρ vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ have $2M\rho$ elements, but there are $0.5\rho(\rho + 1)$ constraints in (4). Therefore the degrees of freedom are given by $M^2 + 2M\rho - 0.5\rho(\rho + 1)$.

A unfactorizable unimodular matrix: Though all first order unimodular matrices (LUTs) are factorizable, there are unfactorizable unimodular matrices with order greater than one. One of such examples is

$$\mathbf{G}(z) = \mathbf{I} + \mathbf{a}\mathbf{b}^\dagger z^{-L},$$

where $\mathbf{b}^\dagger\mathbf{a} = 0$ and $L \geq 2$. One can verify that $\mathbf{G}(z)$ is unimodular and its inverse is given by $\mathbf{I} - \mathbf{a}\mathbf{b}^\dagger z^{-L}$. Its degree is equal to L. In fact we can show that we are not able to extract any degree-one unimodular building block from $\mathbf{G}(z)$. To see this, suppose that we can extract one $\mathbf{D}(z)$ from the right hand side. Then $\mathbf{G}(z) = \mathbf{B}(z)\mathbf{D}(z)$, where $\mathbf{B}(z)$ is a causal unimodular matrix with degree $L-1$. Letting $\mathbf{B}(z) = \mathbf{I}+\mathbf{B}_1 z^{-1}+\dots+\mathbf{B}_{L-1}z^{-L+1}$ and carrying out the multiplication, we find that

$$\begin{aligned}&\mathbf{G}(z) = \mathbf{B}(z)\mathbf{D}(z)\\ &= \mathbf{I} + \dots + (\mathbf{B}_{L-1} + \mathbf{B}_{L-2}\mathbf{u}\mathbf{v}^\dagger)z^{-L+1} + \mathbf{B}_{L-1}\mathbf{u}\mathbf{v}^\dagger z^{-L}.\end{aligned}$$

Comparing the coefficients of the z^{-L+1} and z^{-L} terms, we get

$$\begin{aligned}\mathbf{B}_{L-1} + \mathbf{B}_{L-2}\mathbf{u}\mathbf{v}^\dagger &= \mathbf{0}\\ \mathbf{B}_{L-1}\mathbf{u}\mathbf{v}^\dagger &= \mathbf{a}\mathbf{b}^\dagger \neq \mathbf{0}.\end{aligned}$$

Multiplying $\mathbf{u}$ to the right of the first expression, we get $\mathbf{B}_{L-1}\mathbf{u} = \mathbf{0}$ as $\mathbf{v}^\dagger\mathbf{u} = 0$. This implies that $\mathbf{a}\mathbf{b}^\dagger = \mathbf{0}$, a contradiction! Similarly, we can show that we cannot extract any degree-one block from the left hand side of $\mathbf{G}(z)$.

A different characterization of LUT: In [2], a different degree-one unimodular system is introduced. It has the form

$$\widehat{\mathbf{D}}(z) = \mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + \mathbf{u}\mathbf{v}^\dagger z^{-1}, \quad \mathbf{v}^\dagger\mathbf{u} = 0.$$

Comparing $\widehat{\mathbf{D}}(z)$ with $\mathbf{D}(z)$ in (2), one can verify that $\widehat{\mathbf{D}}(z) = (\mathbf{I} - \mathbf{u}\mathbf{v}^\dagger)\mathbf{D}(z)$. In [2], it is shown that the first-order system $\mathbf{I} - \mathbf{U}\mathbf{V}^\dagger + \mathbf{U}\mathbf{V}^\dagger z^{-1}$ is unimodular if and only if the $\rho \times \rho$ matrix $\mathbf{V}^\dagger\mathbf{U}$ has all the eigenvalues equal to zero. Using an approach similar to the proof of Theorem 2, one can also factorize LUTs into $\widehat{\mathbf{D}}(z)$. It is not difficult to show that $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}$ is a degree-ρ LUT if and only if

$$\mathbf{A}(z) = \mathbf{A}(1)\widehat{\mathbf{D}}_0(z)\widehat{\mathbf{D}}_1(z)\dots\widehat{\mathbf{D}}_{\rho-1}(z),$$

where $\widehat{\mathbf{D}}_k(z) = \mathbf{I} - \mathbf{u}_k\mathbf{v}_k^\dagger + \mathbf{u}_k\mathbf{v}_k^\dagger z^{-1}$, and the vectors $\mathbf{u}_k$ and $\mathbf{v}_k$ satisfy (4).

4. LOT, BOLT, AND LUT

In this section, we will compare three classes of first-order systems, namely Lapped Orthogonal Transform (LOT) [1] [4], BiOrthogonal Lapped Transform (BOLT) [2] and Lapped Unimodular Transform (LUT). All of these transforms are first order matrices with FIR inverses. They can respectively be factorized into the following three different degree-one building blocks:

$$\begin{aligned}\mathbf{B}_k(z) &= \mathbf{I} - \mathbf{v}_k\mathbf{v}_k^\dagger + \mathbf{v}_k\mathbf{v}_k^\dagger z^{-1}, & \mathbf{v}_k^\dagger\mathbf{v}_k &= 1;\\ \mathbf{C}_k(z) &= \mathbf{I} - \mathbf{u}_k\mathbf{v}_k^\dagger + \mathbf{u}_k\mathbf{v}_k^\dagger z^{-1}, & \mathbf{u}_k^\dagger\mathbf{v}_k &= 1;\\ \mathbf{D}_k(z) &= \mathbf{I} - \mathbf{u}_k\mathbf{v}_k^\dagger + \mathbf{u}_k\mathbf{v}_k^\dagger z^{-1}, & \mathbf{u}_k^\dagger\mathbf{v}_k &= 0.\end{aligned}$$

Note that $\mathbf{B}_k(z)$, $\mathbf{C}_k(z)$, and $\mathbf{D}_k(z)$ are respectively degree-one PU, CAFACAFI, and unimodular matrices. Combining the above results and those in [1] [4] [2], we can summarize the factorization theorems for LOT, BOLT and LUT as follows:

Theorem 3 *Consider the first-order degree-ρ system* $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}$. *It is*

1. *an LOT if and only if* $\mathbf{A}(z) = \mathbf{A}(1)\ \mathbf{B}_0(z)\ \mathbf{B}_1(z) \ldots \mathbf{B}_{\rho-1}(z)$, *where the vectors* $\mathbf{v}_k$ *are such that* $[\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{\rho-1}]^\dagger[\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{\rho-1}] = \mathbf{I}_\rho$ *[1] [4];*

2. *an BOLT if and only if* $\mathbf{A}(z) = \mathbf{A}(1)\ \mathbf{C}_0(z)\ \mathbf{C}_1(z) \ldots \mathbf{C}_{\rho-1}(z)$, *where the vectors* $\mathbf{u}_k$ *and* $\mathbf{v}_k$ *are such that* $[\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{\rho-1}]^\dagger[\mathbf{u}_0\ \mathbf{u}_1\ \ldots\ \mathbf{u}_{\rho-1}] = \triangle$ *for some lower triangular matrix with all diagonal elements equal to one [2];*

3. *an LUT if and only if* $\mathbf{A}(z) = \mathbf{A}(1)\ \mathbf{D}_0(z)\ \mathbf{D}_1(z) \ldots \mathbf{D}_{\rho-1}(z)$, *where the vectors* $\mathbf{u}_k$ *(independent) and* $\mathbf{v}_k$ *(independent) are such that* $[\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{\rho-1}]^\dagger[\mathbf{u}_0\ \mathbf{u}_1\ \ldots\ \mathbf{u}_{\rho-1}] = \triangle$ *for some lower triangular matrix with all diagonal elements equal to zero.*

First-order FIR system with FIR inverse: Consider a degree-ρ system $\mathbf{A}(z) = \mathbf{A}_0 + \mathbf{A}_1 z^{-1}$. If $[\det \mathbf{A}(z)] = z^{-L}$ for some integer L, then it has an FIR inverse. It has been shown in [2] that such a system can always be characterized as:

$$\mathbf{A}(z) = \mathbf{A}(1)\Big[\mathbf{I} - \widetilde{\mathbf{U}}\widetilde{\mathbf{V}}^\dagger + \widetilde{\mathbf{U}}\widetilde{\mathbf{V}}^\dagger z^{-1}\Big],$$

where $\widetilde{\mathbf{U}}$ and $\widetilde{\mathbf{V}}$ are $M \times \rho$ matrices such that the eigenvalues of their product $\widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{U}}$ are either one or zero. Using Schur's triangularization theorem, we can find unitary $\mathbf{T}$ such that $\mathbf{T}^\dagger\widetilde{\mathbf{V}}^\dagger\widetilde{\mathbf{U}}\mathbf{T} = \triangle$, where $\triangle$ is a lower triangular matrix with its diagonal elements equal to either one or zero. Let $\mathbf{V} = [\mathbf{v}_0\ \mathbf{v}_1\ \ldots\ \mathbf{v}_{\rho-1}] = \widetilde{\mathbf{V}}\mathbf{T}$ and $\mathbf{U} = [\mathbf{u}_0\ \mathbf{u}_1\ \ldots\ \mathbf{u}_{\rho-1}] = \widetilde{\mathbf{U}}\mathbf{T}$. Then one can verify that the matrix $\mathbf{A}(z)$ can be decomposed as

$$\mathbf{A}(z) = \mathbf{E}_0(z)\mathbf{E}_1(z)\ldots\mathbf{E}_{\rho-1}(z),$$

where

$$\mathbf{E}_k(z) = \begin{cases} \mathbf{B}_k(z) & \text{if } \mathbf{u}_k = \mathbf{v}_k, \\ \mathbf{C}_k(z) & \text{if } \mathbf{v}_k^\dagger\mathbf{u}_k = 1, \\ \mathbf{D}_k(z) & \text{if } \mathbf{v}_k^\dagger\mathbf{u}_k = 0. \end{cases}$$

5. VECTOR DPCM WITH FIR ENCODER AND DECODER

In a vector DPCM, we want to find the matrices $\mathbf{P}_k$ such that the variance of the following prediction error vector is minimized.

$$\mathbf{e}(n) = \mathbf{x}(n) + \mathbf{P}_1\mathbf{x}(n-1) + \ldots + \mathbf{P}_L\mathbf{x}(n-L).$$

The optimal solution is well known and can be obtained by solving the normal equations [8]. The transfer function $\mathbf{P}(z) = \mathbf{I} + \mathbf{P}_1 z^{-1} + \ldots + \mathbf{P}_L z^{-L}$ is known as the prediction error polynomial [8]. At the decoder, we need to implement the inverse $\mathbf{P}^{-1}(z)$. Though it is always stable [8], $\mathbf{P}^{-1}(z)$ is in general IIR. In some applications, FIR systems might be preferred. In this case, one can constrain $\mathbf{P}(z)$ to be unimodular. One way to do this is to assume that $\mathbf{P}(z) = \mathbf{D}_0(z)\mathbf{D}_1(z)\ldots\mathbf{D}_N(z)$. This is in general a loss of generality as there are unfactorizable unimodular systems. Then under the constraint $\mathbf{v}_k^\dagger\mathbf{u}_k = 0$ the vectors $\mathbf{u}_k$ and $\mathbf{v}_k$ can be optimized so that the variance of $\mathbf{e}(n)$ is minimized. In the special case of one-step prediction, $\mathbf{P}(z)$ becomes an LUT and it can always be expressed as in (3). Therefore unlike scalar DPCM, a vector DPCM codec can have FIR encoder and FIR decoder!

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Acknowledgement: The authors would like to thank the DSP Group at Caltech for their interest and comments on the topic.