BLOCK BASED DMT SYSTEMS WITH REDUCED REDUNDANCY

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ABSTRACT

There has been great interest in the design of DMT (discrete multitone) transceivers. An M-band DMT transceiver is called block based if the transmitter and the receiver consist of constant matrices. The commonly used DMT systems are mostly block based, e.g., the DFT based system used in transmission over digital subscriber lines. For an FIR channel of order L, it is known that redundancy of length L enables the receiver to cancel ISI completely. Such a scheme allow us to trade bandwidth for ISI cancellation. In block based DMT (BDMT) systems, the redundancy K is typically chosen to be the same as the order of the channel L. In this paper we will consider BDMT transceiver with redundancy $K \leq L$. With the reduced redundancy better bandwidth efficiency can be obtained as will be demonstrated by examples. Furthermore minimum redundancy for BDMT systems will be derived and the transceivers will be parameterized whenever ISI solutions exist.

1. INTRODUCTION

The DMT (discrete-multitone) systems have been shown to be a very useful technique for transmission over frequency selective channels [1]-[4]. Fig. 1 shows an example of an M-band DMT transceiver over channel $P(z)$ with additive noise $\nu(n)$. The example is the so-called *block based* DMT (BDMT), where the transmitter and the receiver consist of constant matrices. The encoding at the transmitter side and the decoding at the receiver end can be performed blockwise. Usually with proper time domain equalization the channel is modeled as an FIR filter of order L. It is known that for FIR channels, the introduction of certain redundancy allows the receiver to cancel ISI completely. In fact, channel equalization can be performed implicitly using FIR transceivers [1]-[4]. Typically in BDMT systems, the redundancy K is chosen to be the same as the order of the channel L for ISI cancellation. In this paper we will consider BDMT transceiver with redundancy $K \leq L$. With the reduced redundancy, we can obtain better bandwidth efficiency. An example will be given to demonstrate that the BDMT with reduced redundancy requires less transmission power than the traditional BDMT. The advantage is particularly significant for a moderate number of bands. Moreover we will show that the minimum redundancy for BDMT systems is $\lceil L/2 \rceil$, where $\lceil x \rceil$ denotes the smallest integer larger than x. When ISI free solutions of BDMT system with minimum redundancy exist, complete parameterization of the transmitter and receiver will be given.

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2. MATRIX REPRESENTATION OF BDMT SYSTEMS

In this section, we will give the ISI free conditions for BDMT systems in matrix form. Consider Fig. 1, where an M-band block based DMT system is shown. Usually the channel is modeled as an LTI filter $P(z)$ with additive noise $\nu(n)$. Assume that $P(z)$ is an FIR filter of order L (a reasonable assumption after time domain equalization). Let $P(z) = p_0 + p_1 z^{-1} + \cdots p_L z^{-L}$ with $p_0 \neq 0$ and $p_L \neq 0$. With interpolation ratio N and number of bands M as in Fig. 1, there are $K = N - M$ redundant samples for every input block of length M. The $N \times N$ system from $\mathbf{y}(n)$ to $\mathbf{r}(n)$ is an LTI system with transfer matrix $\mathbf{C}(z)$, where $\mathbf{C}(z)$ is a so-called pseudo-circulant matrix [5]. Assuming $N > L$, the channel matrix $\mathbf{C}(z)$ is causal, and of order one and we can write it as,

$$\mathbf{C}(z) = \mathbf{C}_0 + z^{-1}\mathbf{C}_1, \tag{1}$$

where $\mathbf{C}_0$ is an $N \times N$ lower triangular Toeplitz matrix with the first column given by $\begin{pmatrix} p_0 & p_1 & \cdots & p_L & 0 & \cdots & 0 \end{pmatrix}^T$. The matrix $\mathbf{C}_1$ is N by N and it is of the form

$$\mathbf{C}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{C}_\triangle \\ \mathbf{0} & \mathbf{0} \end{pmatrix}, \tag{2}$$

where $\mathbf{C}_\triangle$ is an $L \times L$ upper triangular Toeplitz matrix with the first row given by $\begin{pmatrix} p_L & p_{L-1} & \cdots & p_0 \end{pmatrix}$. The matrix $\mathbf{C}_\triangle$ is nonsingular as $p_L \neq 0$.

The overall transfer function $\mathbf{T}(z)$ of the DMT transceiver is also causal, of first order, $\mathbf{T}(z) = \mathbf{T}_0 + z^{-1}\mathbf{T}_1$, where $\mathbf{T}_0 = \mathbf{S}\mathbf{C}_0\mathbf{G}$, and $\mathbf{T}_1 = \mathbf{S}\mathbf{C}_1\mathbf{G}$. The BDMT is ISI free if,

$$\begin{aligned} \mathbf{S}\mathbf{C}_0\mathbf{G} &= \mathbf{I}, \text{ condition (i)} \\ \text{and } \mathbf{S}\mathbf{C}_1\mathbf{G} &= \mathbf{0}. \text{ condition (ii)} \end{aligned} \tag{3}$$

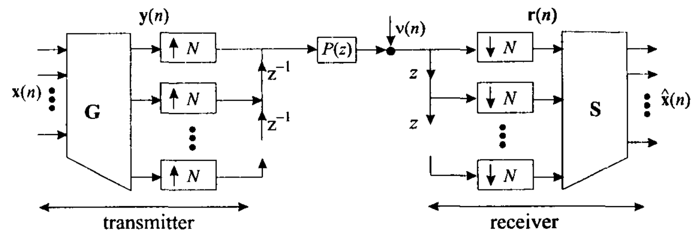

Figure 1: An M-band block based DMT transceiver over channel $P(z)$ with noise $\nu(n)$.

When the second condition holds, the system has zero IBI (inter-block interference), a condition necessary for block-wise encoding and decoding.

3. BDMT TRANSCEIVERS WITH REDUCED REDUNDANCY

The BDMT system can be seen as a special case of FIR transceivers, where the transmitting filters and receiving filters have length $\leq$ the interpolation ratio N. The BDMT transceivers have been studied by a number of researchers [2][3][4]. For a given FIR channel $P(z)$ with order L, redundancy of length $K = L$ is sufficient for the existence of BDMT transceivers.

Two widely used BDMT transceivers. Most of the BDMT transceivers fall into the categories of trailing-zero transmitters and leading-zero receivers. In the DFT based DMT systems [1], redundancy is in the form of cyclic prefix of length L. The prefix is discarded at the receiving end; the receiver is of the leading-zero form, $\mathbf{S}_{LZ} = \begin{pmatrix}\mathbf{0}_{(M\times L)} & \mathbf{S}'_{LZ}\end{pmatrix}$, where $\mathbf{S}'_{LZ}$ is of dimensions $M \times M$. Another commonly used form of redundancy is zero padding. Zero padding of length L are used in [2][3][4]. In this case, the transmitter $\mathbf{G}$ is of the trailing-zero form, $\mathbf{G}_{TZ} = \begin{pmatrix}\mathbf{G}'_{TZ} \\ \mathbf{0}_{(L\times M)}\end{pmatrix}$, where $\mathbf{G}'_{TZ}$ is of dimensions $M \times M$.

When the BDMT has transmitter in the trailing-zero form and receiver in the leading-zero form at the same time, we say the system is in TZ-LZ form. Using TZ-LZ form, ISI free solutions of BDMT with reduced redundancy can be conveniently obtained as we see next.

TZ-LZ BDMT with reduced redundancy

Assume the redundancy is $L/2 \leq K \leq L$. The transmitter is in trailing-zero form and the receiver is in the leading-zeros form given by,

$$\mathbf{G} = \begin{pmatrix}\mathbf{G}' \\ \mathbf{0}_{(K\times M)}\end{pmatrix}, \quad \mathbf{S} = \begin{pmatrix}\mathbf{0}_{(M\times(L-K))} & \mathbf{S}'\end{pmatrix}, \qquad (4)$$

where $\mathbf{G}'$ is $M \times M$ and $\mathbf{S}'$ is $M \times (M + 2K - L)$. Unlike conventional leading-zero receiver, it has only the first $L - K$ columns equal to zeros. In this case, condition (ii) in (3) is satisfies and the ISI free condition reduces to

$$\mathbf{SC}(z)\mathbf{G} = \mathbf{S}'\mathbf{BG}' = \mathbf{I}, \qquad (5)$$

where $\mathbf{B}$ is the bottom left $(M + 2K - L) \times M$ submatrix of $\mathbf{C}(z)$. The matrix $\mathbf{B}$ is Toeplitz; it is given by,

$$\mathbf{B} = \begin{pmatrix} p_{L-K} & \cdots & p_0 & 0 & \cdots & 0 \\ \vdots & \ddots & & & & \\ p_K & & & & & \\ \vdots & \ddots & & \ddots & & p_0 \\ p_L & & & & & \vdots \\ 0 & & & \ddots & & p_{L-K} \\ \vdots & & & & & \vdots \\ 0 & & & 0 & p_L \quad \cdots & p_K \end{pmatrix} \qquad (6)$$

The necessary and sufficient condition for the existence of ISI free transceiver is that the matrix $\mathbf{B}$ has a left inverse. When $K = L/2$ (L even case), $\mathbf{B}$ is M by M and the inverse is unique whenever it exists. If $K > L/2$, the left inverse of B, when it exists, is not unique. For a given $\mathbf{G}'$, we can choose $\mathbf{S}'$ as

$$\mathbf{S}' = \mathbf{G}'^{-1}\mathbf{Q}, \qquad (7)$$

where $\mathbf{Q}$ is any left inverse of $\mathbf{B}$. In most of our experiments, the matrix $\mathbf{B}$ has a left inverse; left inverses of $\mathbf{B}$ do not exist only in some pathological cases.

Example1. *Comparison of ISI free DCT Transceivers with different redundancy.* Consider the channel $P(z)$ and power spectrum of the colored noise $\nu(n)$ shown in Fig. 3(a) and (b). The order of $P(z)$ in this case is 4. These are obtained from a typical DSL environment. Let us consider block based DCT transceivers with two different cases of redundancy, reduced redundancy $K = 3$ and conventional length of redundancy $K = 4$. The transmitter used in this example is as in (4) and $\mathbf{G}'$ is an $M \times M$ DCT matrix. From (5) we know, for an ISI free solution we can choose

$\mathbf{S} = (\mathbf{0} \quad \mathbf{G}'^{-1}\mathbf{Q})$, where $\mathbf{Q} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$ is a left inverse of $\mathbf{B}$. The bits are allocated optimally as in [4]. For a fixed probability of error P_e, and transmission bit rate R_b, the required transmission power $\mathcal{P}(M, K)$ is a function of the number of bands M and redundancy K. We compare the DCT transceiver with $K = 3$ and the DCT transceiver with $K = L = 4$. With $R_b = 3$ bits/sample, Fig. 4 shows the ratio $\mathcal{P}(M, K = 3)/\mathcal{P}(M, K = 4)$ for different values of M. We can see that the DCT transceiver with $K = 3$ performs significantly better than that with $K = 4$, especially for small M.

Minimum redundancy of BDMT transceivers

In what follows, we will consider more general BDMT systems, not restricted to the TZ-LZ form in (4). The transmitter $\mathbf{G}$ is a general $N \times M$ matrix and the receiver $\mathbf{S}$ is a general $M \times N$ matrix. For BDMT systems in the TZ-LZ form, we see that the zero IBI property (condition (ii) of (3)) can be achieved when redundancy $K \geq \lceil L/2 \rceil$. The following lemma will show that $\lceil L/2 \rceil$ is in fact the minimum redundancy for IBI free.

Lemma 1 *Consider the DMT transceiver in Fig. 1 with interpolation ratio N and number of bands M. The DMT system is IBI free, i.e., $\mathbf{SC}_1\mathbf{G} = \mathbf{0}$ only if redundancy K, given by $K = N - M$, satisfies $2K \geq L$.*

Proof: The matrix $\mathbf{C}_1$ is Toeplitz and it has rank L as p_L is assumed to be non zero. Also $\mathbf{G}$ is full rank of dimensions $N \times (N - L)$; the nullity or the dimension of the null space of $\mathbf{G}^T$ is K. We have,

$$rank(\mathbf{C}_1\mathbf{G}) \geq L - K. \tag{8}$$

Equality holds if and only if the null space of $\mathbf{G}^T$ is contained in the row space of $\mathbf{C}_1$. Similarly, the nullity of $\mathbf{S}$ is also K; we have

$$rank(\mathbf{SC}_1\mathbf{G}) \geq rank(\mathbf{C}_1\mathbf{G}) - K \geq L - 2K. \tag{9}$$

The first inequality becomes equality if and only if null space of $\mathbf{S}$ is in the range space of $\mathbf{C}_1\mathbf{G}$. The second inequality is due to (8). When the system is IBI free, we have $rank(\mathbf{SC}_1\mathbf{G}) = 0$ and from (9) we can see that this is true only if $K \geq L/2$. $\Delta\Delta\Delta$

Remarks. For a given N, we can compute the minimum redundancy for the existence of FIR transceivers as in [6]. When the minimum redundancy $> L/2$, FIR solutions do not exist, let alone block based solutions. The condition in Lemma 1 gives only the necessary condition for the existence of IBI free block based transceivers. It does not guarantee existence. The problem of finding the minimum redundancy sufficient for the existence of IBI free BDMT transceivers is still open.

4. PARAMETERIZATION OF BDMT SYSTEMS WITH MINIMUM REDUNDANCY

When ISI free BDMT systems with minimum redundancy exist, we can parameterize the solutions. We will assume that L is even and $K = L/2$. The solutions for odd L can be parameterized in a similar manner. Let

$$\mathbf{S} = (\mathbf{S}_0 \quad \mathbf{S}_1), \quad \mathbf{G} = \begin{pmatrix} \mathbf{G}_0 \\ \mathbf{G}_1 \end{pmatrix},$$

where $\mathbf{S}_0$ and $\mathbf{S}_1$ are of dimensions respectively $M \times L$ and $M \times (M - L/2)$, and $\mathbf{G}_0$ $\mathbf{G}_1$ are of dimensions respectively $(M - L/2) \times M$ and $L \times M$.

Lemma 2 *Consider the BDMT transceiver with redundancy $K = L/2$, where L is even. (a) The DMT system is ISI free only if $rank(\mathbf{S}_0) = rank(\mathbf{G}_1) = L/2$. (b) The transceivers satisfying these rank conditions in (a) are of the form*

$$\begin{aligned} \mathbf{S} &= \mathbf{S}_M \underbrace{\begin{pmatrix} \mathbf{\Phi}_S & \mathbf{I}_M \\ \mathbf{0} & \end{pmatrix} \begin{pmatrix} \mathbf{P}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-L/2} \end{pmatrix}}_{\mathbf{A}_S}, \\ \mathbf{G} &= \underbrace{\begin{pmatrix} \mathbf{I}_{M-L/2} & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_G \end{pmatrix} \begin{pmatrix} & \mathbf{I}_M \\ \mathbf{0} & \mathbf{\Phi}_G \end{pmatrix}}_{\mathbf{A}_G} \mathbf{G}_M, \end{aligned} \tag{10}$$

where $\mathbf{\Phi}_S$ and $\mathbf{\Phi}_G$ are $L/2$ by $L/2$ arbitrary matrices and $\mathbf{P}_S$ and $\mathbf{P}_G$ are $L \times L$ permutation matrices.

The proof of Lemma 2 is given in [6]. Note that the matrices $\mathbf{S}_M$ and $\mathbf{G}_M$ are M by M and they are nonsingular because $\mathbf{S}$ and $\mathbf{G}$ have full rank. Using (10), condition (ii) in (3) becomes,

$$(\mathbf{\Phi}_S \quad \mathbf{I}_{L/2})\,\mathbf{P}_S\mathbf{C}_\Delta\mathbf{P}_G \begin{pmatrix} \mathbf{I}_{L/2} \\ \mathbf{\Phi}_G \end{pmatrix} = \mathbf{0}. \tag{11}$$

Let

$$\mathbf{P}_S\mathbf{C}_\Delta\mathbf{P}_G = \begin{pmatrix} \mathbf{\Delta}_{00} & \mathbf{\Delta}_{01} \\ \mathbf{\Delta}_{10} & \mathbf{\Delta}_{11} \end{pmatrix},$$

then (11) can be rewritten as

$$\mathbf{\Phi}_S\mathbf{\Delta}_{00} + \mathbf{\Phi}_S\mathbf{\Delta}_{01}\mathbf{\Phi}_G + \mathbf{\Delta}_{10} + \mathbf{\Delta}_{11}\mathbf{\Phi}_G = \mathbf{0}. \tag{12}$$

Using $\mathbf{G}$ and $\mathbf{S}$ in (10), condition (i) in (3) becomes

$$\mathbf{S}_M \underbrace{\mathbf{A}_S\mathbf{C}_0\mathbf{A}_G}_{\mathbf{C}_M} \mathbf{G}_M = \mathbf{I}. \tag{13}$$

Using (10) we have converted the two conditions in (3) to (12) and (13). From (12) and (13), we can solve for the receiver when the transmitter is given and similarly we can solve for the transmitter when the receiver is given. For example, suppose the transmitter is given, that is, $\mathbf{\Phi}_G$ and

$\mathbf{G}_M$ are given. We can solve for $\mathbf{\Phi}_S$ in (12). In particular, if $\mathbf{\Delta}_{00} + \mathbf{\Delta}_{01}\mathbf{\Phi}_G$ is nonsingular, we have

$$\mathbf{\Phi}_S = -\left(\mathbf{\Delta}_{10} + \mathbf{\Delta}_{11}\mathbf{\Phi}_G\right)\left(\mathbf{\Delta}_{00} + \mathbf{\Delta}_{01}\mathbf{\Phi}_G\right)^{-1}. \quad (14)$$

Eq. (13) can be satisfied if $\mathbf{C}_M$ is nonsingular. In this case, $\mathbf{S}_M = \mathbf{G}_M^{-1}\mathbf{C}_M^{-1}$.

The design procedure can be summarized as follows. Consider the case when the transmitter is given. Choose $\mathbf{G}_M$, $\mathbf{\Phi}_G$ and $\mathbf{P}_G$ for the transmitter in (10) and also choose $\mathbf{P}_S$ for the receiver. The matrix $\mathbf{G}_M$ is an arbitrary $M \times M$ nonsingular matrix and, $\mathbf{P}_G$ and $\mathbf{P}_S$ are arbitrary permutation matrices. We can solve for $\mathbf{\Phi}_S$ according to (14). Form the matrix $\mathbf{C}_M$ in (13) and compute $\mathbf{S}_M = \mathbf{G}_M^{-1}\mathbf{C}_M^{-1}$. For the case when the receiver is given, the design procedure is similar.

In the parameterization, no additional assumption has been made on the transmitter matrix and the receiver matrix except that they achieve zero ISI. Therefore, whenever BDMT with redundancy $K = L/2$ exists, it can be parameterized as in this section. The parameterization is useful in some pathological cases where ISI free BDMT solutions exist but there are no ISI free TZ-LZ solutions. One such example is given below.

Example2. Consider the FIR channel $P(z) = (1 - z^{-2})^3$ with order $L = 6$. Let $M = 5$ and $K = 3$, then we have $N = M + K = 8$. We can verify that in this case the matrix $\mathbf{B}$ given in (6) is singular. There are no ISI free solutions for BDMT in the TZ-LZ form. On the other hand, let us choose

$$\mathbf{\Phi}_G = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We can verify that the matrix $\mathbf{\Delta}_{00} + \mathbf{\Delta}_{01}\mathbf{\Phi}_G$ is nonsingular and the matrix $\mathbf{C}_M$ in (13) is also nonsingular. We can obtain solution of $\mathbf{\Phi}_S$ from (14) and $\mathbf{S}_M = \mathbf{G}_M^{-1}\mathbf{C}_M^{-1}$ for arbitrarily chosen nonsingular $\mathbf{G}_M$.

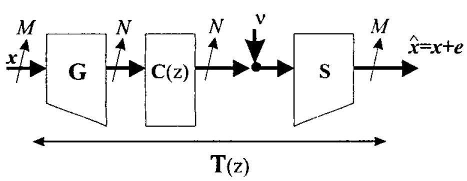

Figure 2: Matrix representation of the block based DMT transceiver.

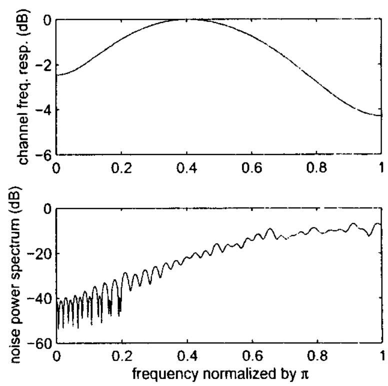

Figure 3: (a) The magnitude response of the channel $P(z)$; (b) The power spectrum of the additive noise $\nu(n)$.

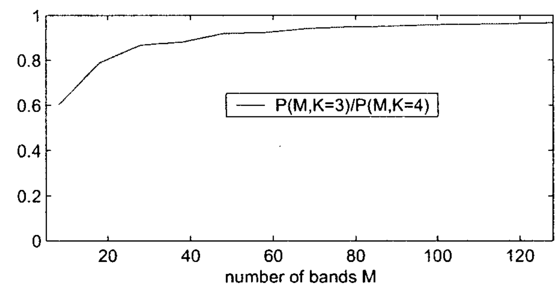

Figure 4: The ratio of the transmission power of the DCT transceiver with reduced redundancy over that with conventional length of redundancy, $\mathcal{P}(M, K = 3)/\mathcal{P}(M, K = 4)$, for transmission bit rate $R_b = 3$ bits/sample.

5. REFERENCES

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