Optimal Transform for Colored Noise Suppression+

*See-May Phoong**
Dept. of EE & Inst. of Comm Engr.
National Taiwan Univ., China

Yuan-Pei Lin
Dept. Elect. and Control Engr.
National Chiao Tung Univ. China

Abstract: There has been interest in the application of filter banks and transforms to noise suppression. Recently, Akkarakaran and Vaidyanathan showed that when the noise is white, the optimal transform (not necessary unitary) for noise suppression is the Karhunen-Loeve transform (KLT). Moreover they showed that for the case of colored noise, if both the noise and signal have a common KLT, then the KLT is optimal for subband noise suppression. In this paper, we will derive the optimal transform (not restricting to the class of unitary transforms) for the noise suppression problem when the signal and noise have arbitrary spectrum.

Keywords: filter bank, transform, denoising, noise suppression.

I. Introduction

In recent years, there has been considerable interest in the application of filter bank (FB) to noise suppression (denoising) (see [1-6] and references therein). Both maximally decimated FBs [1, 2] and under decimated FBs [3-5] have been considered for such an application. While the undecimated FBs [3] or the dual-tree discrete wavelet transforms [4, 5] have the advantage of being perfect or nearly shift-invariant, they correspond to overcomplete expansion. In this paper, we study the maximally decimated case only. Fig. 1 shows such a FB-based noise reduction scheme. The black boxes in the figure denote the subband denoising operations.

+Supported in parts by National Science Council, Taiwan, R.O.C., under contract numbers NSC 90-2213-E-002-097 and 90-2213-E-009-108, and Ministry of Education, Taiwan, R.O.C., under contract number 89-E-FA06-2-4, and the Lee and MTI Center or Networking Research.

*Corresponding author, E-mail: smp@cc.ee.ntu.edu.tw

There are various subband denoising schemes such as the Wiener filtering, soft thresholding, hard thresholding, input adaptive thresholding, etc. In [6], a practical thresholding scheme that is applied to each subband sample was proposed. It was shown [6] that the proposed thresholding scheme, which thresholds the coefficients to a specific level, provides a quasi-optimal min-max estimator of a noisy piecewise-smooth signals.

Unlike [6], our goal is to design the FB-based denoising scheme to minimize the output error variance

$$\mathcal{E}_n = E\{(\mathbf{y}(n) - \mathbf{s}(n))^T(\mathbf{y}(n) - \mathbf{s}(n))\}, \quad (1)$$

where s(n) is the desired signal and y(n) is the output signal, as shown in Fig. 1. We will consider the case when the subband denoising operation is carried out by multiplication with a set of constants. Recently Akkarakaran and Vaidyanathan [1] showed that for the white noise case the principle component FB (PCFB) (if exists) is the optimal orthonormal FB for noise suppression. Moreover the optimality of PCFB holds even when any combination of Wiener filter or hard threshold is used in the subband. For the special case of memoryless transform where **T** is a constant matrix, PCFB reduces to the well-known KLT. If the additive noise is colored, the only solution known is for the case when the autocorrelation matrices of s(n) and v(n) have a common KLT. For this restrictive case, the common KLT is showed [1] to be the optimal memoryless transform when zeroth-order Wiener filter is applied in the subband.

In this paper, we consider the more general case of arbitrary signal and noise spectrum. For this general case, we will first show that this seemingly difficult optimization problem can be easily solved using Wiener theory. The optimal memoryless transform (not restricting to unitary transform) for colored noise suppression is given in closed form. The optimal transform decorrelates both the signal s(n) and the noise v(n). It can be realized as a cascade of a signal (or noise) whitening matrix followed by a noise (correspondingly signal) decorrelating matrix. Simulation shows that the optimal biorthogonal transform outperforms the orthogonal transforms that decorrelate the signal or noise.

II. Optimal Transform for Denoising

Let the input vector **x**(n)=**s**(n)+v(n), where the desired signal is **s**(n) and the additive noise is v(n). Assume that **s**(n) and v(n) are real zero-mean WSS uncorrelated vector processes. The $M\times M$ autocorrelation matrices of **x**(n), **s**(n) and v(n) are denoted by $\mathbf{R}_x$, $\mathbf{R}_s$ and $\mathbf{R}_v$ respectively. As **s**(n) and v(n) are uncorrelated, we have

$$\mathbf{R}_x = \mathbf{R}_s + \mathbf{R}_v.$$

Without much loss of generality, we assume that the matrix $\mathbf{R}_x$ is invertible. Hence $\mathbf{R}_x$ is positive definite. Let $\mathbf{R}_x^{1/2}$ be the unique positive definite matrix that satisfies [7]

$$\mathbf{R}_x^{1/2}\,\mathbf{R}_x^{1/2}= \mathbf{R}_x$$

We consider only the class of FB with constant polyphase matrices. That is, the matrix **T** is a nonsingular constant matrix. Assume that the subband operation is carried out by multiplication with a set of constants k_i Therefore the subband operations denoted by the black boxes can bewritten as the diagonal matrix

$$\mathbf{K} = \begin{bmatrix} k_0 & 0 & \cdots & 0 \\ 0 & k_1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & k_{M-1} \end{bmatrix}. \qquad (2)$$

Our aim is to find the best transform **T** and the optimal k_i such that the output error variance ε_n n in (1) is minimized. In the following, we will show that such a seemingly difficult optimization problem can be easily solved using Wiener theory.

Note that the transfer function from x(n) to y(n) is the constant matrix $\mathbf{P}=\mathbf{T}\mathbf{K}\mathbf{T}^{-1}$. From Wiener theory, we know that the output error variance ε_n is lower bounded by that obtained by the Wiener filter. This lower bound is achieved if and only if the matrix **P** is the Wiener filter. For an input with desired signal s(n) and noise v(n), it is known that the Wiener filter is given by:

$$\mathbf{P}_{wiener} = \mathbf{R}_s\mathbf{R}_x^{-1}. \qquad (3)$$

The lower bound on ε_n is given by

$$\varepsilon_{min} = tr[\mathbf{R}_s - \mathbf{R}_s\mathbf{R}_x^{-1}\mathbf{R}_s]. \qquad (4)$$

Observe that, if $\mathbf{P}_{weiner}$ is diagonalizable, then we can achieve this lower bound by choosing the columns of **T** be the eigenvectors of $\mathbf{P}_{weiner}$ and k_i to be the eigenvalue of $\mathbf{P}_{weiner}$. To show $\mathbf{P}_{wiener}$ is always diagonalizable, we rewrite $\mathbf{P}_{weiner}$ as:

$$\begin{aligned}\mathbf{P}_{wiener} &= \mathbf{R}_x^{1/2}\underbrace{[\mathbf{R}_x^{-1/2}\mathbf{R}_s\mathbf{R}_x^{-1/2}]}_{\mathbf{A}}\mathbf{R}_x^{-1/2} \\ &= \mathbf{R}_x^{1/2}\mathbf{Q}\mathbf{D}\mathbf{Q}^{t}\mathbf{R}_x^{-1/2}.\end{aligned}$$

As the matrix **A** in the above equation is symmetric, there exists a unitary matrix **Q** such that $\mathbf{A}=\mathbf{Q}\mathbf{D}\mathbf{Q}^{+}$ for some diagonal matrix **D**. Hence one choice of the optimal transform is

$$\mathbf{T}_{opt} = \mathbf{R}_x^{1/2}\mathbf{Q}. \qquad (5)$$

From Fig. 1, we see that the subband signal is given by $\mathbf{T}_{opt}^{-1}\mathbf{x}(n)=\mathbf{Q}^{+}\mathbf{R}_x^{-1/2}\mathbf{x}(n)$. The optimal $\mathbf{T}_{opt}^{-1}$ performs two tasks: the matrix $\mathbf{R}_x^{-1/2}$ whitens the input **x**(n) while the unitary matrix $\mathbf{Q}^{+}$ decorrelates the filtered desired signal $\mathbf{R}_x^{-1/2}$s(n).

The optimal k_i are the eigenvalues of the Wiener filter P_{weiner}. Since eigenvalues are unique, the best k_i are unique. In fact one can show that this optimal subband denoising matrix $\mathbf{K}_{opt}=\mathbf{D}$ is the Wiener filter for the subband signal $\tilde{\mathbf{s}}(n)$ plus noise $\tilde{\mathbf{v}}(n)$. To see this, we consider Fig. 1.

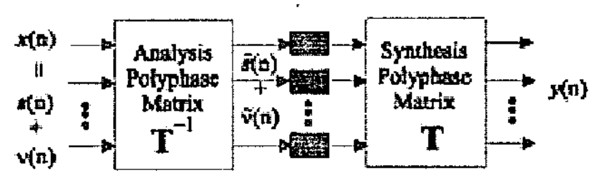

Figure 1: A filter bank based noise suppression scheme.

When the transform is chosen optimally as in (5), the subband signals are given by

$$\tilde{s}(n) = \mathbf{Q}^{\dagger}\mathbf{R}_x^{-1/2}s(n), \quad \tilde{\nu}(n) = \mathbf{Q}^{\dagger}\mathbf{R}_x^{-1/2}\nu(n).$$

Their corresponding autocorrelation matrices are given by

$$\mathbf{R}_{\tilde{s}} = \mathbf{Q}^{\dagger}\mathbf{R}_x^{-1/2}\mathbf{R}_s\mathbf{R}_x^{-1/2}\mathbf{Q} = \mathbf{D},$$

$$\mathbf{R}_{\tilde{\nu}} = \mathbf{Q}^{\dagger}\mathbf{R}_x^{-1/2}\mathbf{R}_\nu\mathbf{R}_x^{-1/2}\mathbf{Q} = \mathbf{I} - \mathbf{D}.$$

In other words, both $\tilde{\mathbf{s}}(n)$ and $\tilde{\mathbf{v}}(n)$ are uncorrelated. Using the above expressions, one immediately sees that the Wiener filter for $\tilde{\mathbf{s}}(n)$ plus $\tilde{\mathbf{v}}(n)$ is given by **D**, which is equal to $\mathbf{K}_{opt}$. Summarizing the results, we have

Theorem 1 *Consider the denoising scheme in Fig. 1. Suppose that the subband operation is taken as (2). Assume that R_x is nonsingular. Then the optimal transform that minimizes $\mathcal{E}_n$ in (1) is given by $T_{opt} = R_x^{1/2} Q$, where Q is a unitary matrix that diagonalizes the matrix ($R_x^{-1/2}$ R_s $R_x^{-1/2}$). The minimized output error variance is given by $\mathcal{E}_{min}$ in (4).*

Cases when R_s or R_v is nonsingular: Assume that $\mathbf{R}_s$ is nonsingular. We can rewrite the Wiener solution in (3) as

$$\mathbf{P}_{wiener} = \mathbf{R}_s^{1/2}(\mathbf{I} + \mathbf{R}_s^{-1/2}\mathbf{R}_\nu\mathbf{R}_s^{-1/2})^{-1}\mathbf{R}_s^{-1/2}.$$

Let Q_0 be a unitary matrix such that $\mathbf{R}_s^{-1/2}\mathbf{R}_v\mathbf{R}_s^{-1/2} = \mathbf{Q}_0\mathbf{D}_0\mathbf{Q}_0^+$ for some diagonal matrix $\mathbf{D}_0$. One can verify that the optimal transform can also be expressed as

$$\mathbf{T}_{opt} = \mathbf{R}_s^{1/2}\mathbf{Q}_0. \qquad (6)$$

The optimal **K** is given by $\mathbf{K}_{opt} = (\mathbf{I}+\mathbf{D}_0)^{-1}$. It is not difficult to verify that $\mathbf{K}_{opt}$ is the Wiener filter for the subband signal $\tilde{\mathbf{s}}(n)$ plus noise $\tilde{\mathbf{v}}(n)$. On the other hand, if $\mathbf{R}_v$ is nonsingular. Following a similar approach, one can verify that the optimal transform can be expressed as:

$$\mathbf{T}_{opt} = \mathbf{R}_\nu^{1/2}\mathbf{Q}_1, \qquad (7)$$

where $\mathbf{Q}_1$ is a unitary matrix such that $\mathbf{R}_s^{-1/2}\mathbf{R}_v\mathbf{R}_s^{-1/2} = \mathbf{Q}_1\mathbf{D}_1\mathbf{Q}_1^+$ for some diagonal matrix $\mathbf{D}_1$. In this case, one can show that the optimal subband operation is the Wiener filter for its input and it is given by $\mathbf{K}_{opt}=\mathbf{D}_1(\mathbf{D}_1+\mathbf{I})^{-1}$.

Three interpretations of the optimal transforms: From Theorem 1, equations(6) and (7), we obtain three different expressions for the optimal $\mathbf{T}_{opt}^{-1}$: (1)$\mathbf{T}_{opt}^{-1}=\mathbf{Q}^+\mathbf{R}_x^{-1/2}$, where $\mathbf{Q}^+$ diagonalizes ($\mathbf{R}_x^{-1/2}$ $\mathbf{R}_s$ $\mathbf{R}_x^{-1/2}$); the optimal transform is a cascade of an input (signal plus noise) whitener followed by a signal decorrelator. The matrix $\mathbf{R}_x^{-1/2}$ whitens the input $\mathbf{x}$(n) while $\mathbf{Q}^+$ decorrelates the filtered signal $\mathbf{R}_x^{-1/2}\mathbf{s}$(n). (2) $\mathbf{T}_{opt}^{-1}=\mathbf{Q}_0^+\mathbf{R}_s^{-1/2}$, where $\mathbf{Q}_0^+$ diagonalizes (($\mathbf{R}_s^{-1/2}$ $\mathbf{R}_v$ $\mathbf{R}_s^{-1/2}$); the optimal transform is a cascade of a signal whitener followed by a noise decorrelator. The matrix $\mathbf{R}_s^{-1/2}$ whitens the signal s(n) while $\mathbf{Q}^+$ decorrelates the filtered noise $\mathbf{R}_s^{-1/2}$v(n). (3) $\mathbf{T}_{opt}^{-1}=\mathbf{Q}_1^+\mathbf{R}_v^{-1/2}$, where $\mathbf{Q}_1^+$ diagonalizes ($\mathbf{R}_v^{-1/2}\mathbf{R}_\sigma\mathbf{R}_v^{-1/2}$); the optimal transform is a cascade of a noise whitener followed by a signal decorrelator. The matrix $\mathbf{R}_n^{-1/2}$ whitens the noise v(n) while $\mathbf{Q}_1^+$ decorrelates the filtered signal $\mathbf{R}_n^{-1/2}$s(n).

Remarks and discussions:

1. In the case that $\mathbf{R}_s$ and $\mathbf{R}_v$ have a common KLT, both $\mathbf{R}_s$ and **Rv** can be simultaneously diagonalized by the same unitary matrix **Q**. The optimal transform can simply be chosen as $\mathbf{Q}^+$. Thus our solution reduces to that given in Theorem 7 of (1).
2. When the noise (or the desired signal) is white, then**Rv=I** (or $\mathbf{R}_s$=**I**). This becomes a special case of common KLT.
3. The above results can be generalized to the case of unconstrained filter length, where the transform and the subband Wiener filters are allowed to be ideal filters. The entire proof and derivations carry through by simply replacing the correlation matrices by the power spectral matrices.

4. For the case of FIR matrix **T**(z), the optimal solution is still an open problem.

III. Simulation

In this section, we compare the performance of the optimal transform and the KLT for colored noise suppression. The dimension of the transform is M=8. The vectors s(n) and v(n) are respectively the blocked versions of scalar uncorrelated WSS processes s(n) and v(n). The signal s(n) is an AR(1) process with correlation coefficient $\rho_s^{|k|}$. The noise v(n) is an AR(1) process with correlation coefficient of $\rho_s^{|k|}$. We compare the output error variances of the following two cases: (i) $\varepsilon_{\min}$ in Theorem 1; and (ii) ε_{klt} : the output error variance when **T** is the KLT for s(n) and k_i is taken as the zeroth order Wiener filter for its input (this is the optimal transform if the noise were white). Fig. 2 shows the results for $0.7 \le \rho_s \le 0.99$ and and ρ_v=-0.7. As we can see, $\varepsilon_{\min}$ is smaller than ε_{klt} and the gain decreases when ρ_s increases. This is because when ρ_s is nearly 1, a large portion of the noise reduction can be obtained by exploiting the correlation of the signal alone. Fig. 3 shows the results for $-0.7 \le \rho_s \le 0$ and ρ_s=0.7. From the figure, we see that as ρ_v decreases to 0, the two curves converge. When ρ_v=0, the noise is white and in this case the optimal transform reduces to the KLT of s(n).

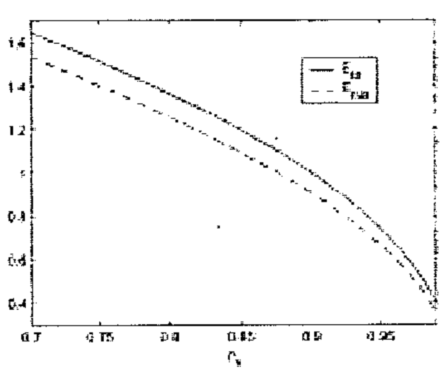

Figure 2: Output error variances when $0.7 \le \rho_s \le 0.99$ and $\rho_v = -0.7$.

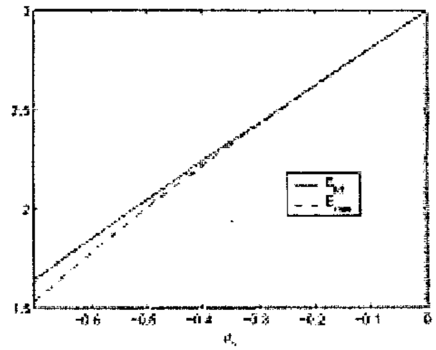

Figure 3: Output error variances when $\rho_s = 0.7$ and $-0.7 \le \rho_v \le 0$.

IV. Conclusions

In this paper, we derive the optimal transform for noise suppression with subband Wiener filtering when both the signal and noise are colored. The optimal transform is formed by the eigenvectors of an associated Wiener matrix and it can be realized as a cascade of a signal (or noise) whitening matrix followed by a noise (correspondingly signal) decorrelating matrix. Simulation shows that the optimal biorthogonal transform outperforms the orthogonal transforms that decorrelate the signal or noise.

Acknowledgement:

The authors would like to thank Dr. S. Akkarakaran of Sequoia Communications and Prof. P. P. Vaidyanathan of California Institute of Technology for their discussion and suggestions that give rise to the current simpler derivation of the optimal transform and the generalization to the unconstrained case in Remark 3.

References

[1] S. Akkarakaran and P. P. Vaidyanathan, ``Filterbank optimization with convex objectives and the optimality of principle component forms," *IEEE Trans. Signal Proc.*, pp. 100-114, Jan. 2001.

[2] S. Akkarakaran and P. P. Vaidyanathan, ``Results on principle component filter banks: Colored noise suppression and existence issues," *IEEE Trans. Info. Theory*, pp. 1003-1020, March 2001.

[3] R. R. Coifman and D. L. Donoho, Translation-invariant denoising, in A. Antoniadis, editor, *Wavelets and*

Statistics. Springer-Verlag Lecture Notes, 1995.
[4] I. W. Selesnick, ``The design of Hilbert transform pairs of wavelet bases via the flat delay filter" *in Proc. IEEE Intl. Conf. on Acoustics, Speech, and Signal Processing,* 2001.
[5] N. G Kingsbury, ``Shift invariant properties of the dual tree complex wavelet transform," *in Proc. IEEE Intl. Conf. on Acoustics, Speech, and Signal Processing,* 1999.
[6] D. L. Donoho and I. M. Johnstone, ``Ideal spatial adaptation by wavelet shrinkage," Biometrika, pp. 425-455, 1994.
[7] R. A. Horn, and C. R. *Johnson, Matrix analysis,* Cambridge University Press, 1985