Two-Dimensional Four-Parallelogram Filter Banks

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Abstract

The most commonly used two-dimensional filter banks are separable filter banks, which are obtained by cascading two one-dimensional (1D) filter banks in the form of a tree. The supports of the analysis and synthesis filters in the separable systems are the union of four rectangles. The natural nonseparable generalization of of such supports are the supports that are the union of four parallelograms. In this paper, we study the class of filter banks in which the supports of the analysis and synthesis filters are four parallelograms. We will study various types of support configurations for the four-parallelogram filter banks. Conditions on the configurations will be derived such that good design of analysis and synthesis filters are possible.

1. Introduction

Recently, there has been considerable interest in the design of two-dimensional (2D) maximally decimated filter banks (Fig. 1), [1]-[4]. For example, perfect reconstruction is achieved in [1] for a 2D two-channel FIR filter bank with diamond-shaped filters. A nonseparable generalization for 2D CMFB is considered in [4]. In [2], several issues regarding design of multidimensional filter bank are treated. In particular, the concept of support permissibility is introduced and discussed from a pictorial viewpoint. It is argued therein that with certain support configurations in a filter bank, a considerable amount of aliasing will remain uncancelled if the individual filters have good attenuation. In this case, the support configuration for the analysis and synthesis filter is called nonpermissible. The one-dimensional (1D) uniform DFT filter bank [3] is known to be an example of nonpermissible nature.

The concept of support permissibility was further exploited in [5] and [6] to design the two parallelogram filter banks. That is the class of systems in which the supports of the analysis filters consist of two parallelograms, each a shifted version of a parallelogram prototype. Fig. 2 (a) and (b) show respectively a parallelogram prototype and the support (passband) of a typical analysis filter in a two parallelogram filter bank. It is mentioned in [6] that for successful design of the analysis and synthesis filters, the configuration of the filter bank should be aliasfree(M). Namely, when the analysis filters are ideal filters, the outputs of the analysis filters allows aliasfree M-fold decimation.

In practical cases, non-ideal roll-off of the filter causes aliasing in the subband. When the analysis filters are decimated and then expanded by M, each has $J(\mathbf{M})-1$ images. All the images are attenuated to the stopband level of the synthesis filters except those images that are adjacent to the passband of the synthesis filters. These adjacent images result in different major aliasing depending on the type of adjacency involved. For example, in Fig. 3(a) one image of the analysis filter $H_k(\boldsymbol{\omega})$ is edge adjacent to the synthesis filter $F_k(\boldsymbol{\omega})$ and results in edge aliasing and similarly the image in Fig. 3(b) results in vertex aliasing.

When the configurations are not constructed properly, it is possible that some edge aliasing can not be canceled if the analysis and synthesis filters have good frequency selectivity. Such configurations are called edge nonpermissible. Similarly, if some vertex aliasing in a configuration are uncancelable when the filters have good frequency selectivity, the configuration is called vertex nonpermissible. So for the individual filters to have good frequency selectivity, it is necessary that the configuration have permissibility [2],[4], which includes edge and vertex permissibility [6]. In this case, the importance of edge permissibility is much greater than vertex permissibility. Although the two-parallelogram filter banks can not be both edge and vertex permissible in general, the two-parallelogram filter banks can have edge permissibility. Edge permissible two-parallelogram cosine modulated filter banks are constructed and designed in [6].

In this paper, we study four parallelogram filter banks, the class of systems in which the support of each analysis filter consists of four parallelograms as shown in Fig. 4. We will discuss permissibility of possible configurations for the four parallelogram filter banks. The conditions on the configurations that ensure permissibility will be derived.

Notations. All notations in this paper are as in [3]. In particular, we will use the following notations.

1. The Fourier transformation of a 2D signal $x(\mathbf{n})$ will be denoted by $X(\boldsymbol{\omega})$.
2. The notation $\mathbf{I}$ denotes a 2×2 identity matrix.
3. *Unimodular matrix*: An integer matrix $\mathbf{U}$ is unimodular if $|\det \mathbf{U}|=1$.
4. *The* $\mathcal{N}(\mathbf{M})$ *notation*: Let $\mathbf{M}$ be a nonsingular integer matrix. The notation $\mathcal{N}(\mathbf{M})$ is defined as the set containing integer vectors of the form

$$\mathbf{n}=\mathbf{M}\mathbf{x}, \mathbf{x}\in[0,1)^2.$$

The number of elements in $\mathcal{N}(\mathbf{M})$ is denoted by $J(\mathbf{M})$, which is equal to $|\det \mathbf{M}|$.

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5. The symmetric parallelepiped $SPD(\mathbf{M})$ is the set

$$SPD(\mathbf{M}) = \{\mathbf{Mx}, \mathbf{x} \in [-1,1)^2\}.$$

2. Construction of Four Parallelogram Filter Banks

The simplest way to design a four-parallelogram filter banks is using separable filter banks. A separable 2D filter bank can be obtained by cascading two 1D filter banks in the form of tree structures. The resulting 2D analysis and synthesis filters are product of two 1D filters and are separable; the support of each analysis filter is the union of four rectangles (Fig. 5). The 2D analysis and synthesis filters can have good frequency selectivity if the 1D filters have good frequency selectivity. Therefore the separable 2D filter banks are both edge and vertex permissible. In this paper we will pursue both edge and vertex permissibility for the four-parallelogram filter banks.

Consider the 2D filter bank with decimation matrix $\mathbf{M}$ in Fig. 1. To obtain a support configuration for the four parallelogram filter bank, first we construct a parallelogram prototype. Then we shift the prototype properly and combine four shifted parallelograms to obtain the support for the analysis filters. The synthesis filters usually have the same supports as the corresponding analysis filters. As the support of each analysis filter consists of four shifted versions of the parallelogram prototype, the area of the parallelogram prototype is one fourth the area of an analysis filter support. In a $J(\mathbf{M})$ channel filter bank, the area of an analysis filter support is usually the same as $SPD(\pi\mathbf{M}^{-T})$. We can describe the parallelogram prototype as $SPD(\pi\mathbf{N}^{-T})$, where $\mathbf{N} = \mathbf{ML}$, for some integer matrix $\mathbf{L}$ with $|\det \mathbf{L}| = 4$. Since $|\det \mathbf{N}| = 4|\det \mathbf{M}|$, the area of $SPD(\pi\mathbf{N}^{-T})$ is one fourth the area of $SPD(\pi\mathbf{M}^{-T})$. For a given decimation matrix $\mathbf{M}$, the choice of $\mathbf{L}$ will determine the parallelogram prototype and indirectly affect the support of the analysis filters. Having constructed the parallelogram prototype, we combine four shifted versions of the parallelogram prototype to obtain the supports of the analysis filters. However, there are a variety of possible configurations for the four-parallelogram filter banks. For example, consider the lowpass analysis filter $H_0(\boldsymbol{\omega})$. Fig. 6 (a) and (b) show two of the possible supports for $H_0(\boldsymbol{\omega})$. The support of $H_0(\boldsymbol{\omega})$ is different when the four parallelograms in the support of $H_0(\boldsymbol{\omega})$ are glued in different manners.

The preceding construction indicates that for a given decimation matrix $\mathbf{M}$, the support configuration is determined by the following two steps. Step one, choose $\mathbf{L}$ and hence the parallelogram prototype. Step two, shift the parallelogram prototype properly and combine four shifted copies to obtain the support for each analysis filter. We will construct various types of configurations for the four-parallelogram filter banks and discuss permissibility of these configurations.

3. The Simplistic Four Parallelogram Filter Banks

In this section, we consider a special type of four-parallelogram filter banks, which will be called the simplistic four-parallelogram filter banks. There are two features associated with this class of systems.

1. For a given decimation matrix $\mathbf{M}$, the matrix $\mathbf{N}$ that determines the parallelogram prototype $SPD(\pi\mathbf{N}^{-T})$ is given by $\mathbf{N} = \mathbf{ML}$ with $\mathbf{L} = 2\mathbf{I}$.
2. The support of the lowpass analysis filter $H_0(\boldsymbol{\omega})$ is the parallelogram $SPD(\pi\mathbf{M}^{-T})$, which is a natural generalization of the lowpass analysis filter of separable filter banks.

For example, let $\mathbf{M} = \begin{bmatrix} 8 & -5 \\ -4 & 5 \end{bmatrix}$ then the parallelogram prototype $SPD(\pi\mathbf{N}^{-T})$ and the support of the lowpass analysis filter $SPD(\pi\mathbf{M}^{-T})$ are as shown in Fig. 7.

Frequency normalization. In Fig. 7, the support of $H_0(\boldsymbol{\omega})$ is shown with ω_0 and ω_1 as two axes. For convenience, we will normalize the frequency plane by $2\pi\mathbf{N}^{-T}$; we define $\boldsymbol{\nu} = 2\pi\mathbf{N}^{-T}\boldsymbol{\omega}$ and use ν_0 and ν_1 as the two axes. On the normalized frequency plane, the support of $H_0(\boldsymbol{\omega})$ become the square $SPD(\mathbf{I})$ in Fig. 8. With this axis normalization, the support of $H_0(\boldsymbol{\omega})$ will always appear as $SPD(\mathbf{I})$ for any parallelogram prototype. The support of $H_0(\boldsymbol{\omega})$ is the union of the four smaller squares

$$S_{(0,0)}, S_{(-1,0)}, S_{(0,-1)}, \text{ and } S_{(-1,-1)}. \qquad (1)$$

In Fig. 8 the notation $S_{(k_0,k_1)}$ denotes the square

$$SPD(0.5\mathbf{I}) + \begin{bmatrix} k_0 + 0.5 \\ k_1 + 0.5 \end{bmatrix}.$$

The vector subscript of S should be interpreted modulo $\mathbf{N}^T$. Notice that to have real-coefficient filters, whenever $S_{(k_0,k_1)}$ belongs to a certain analysis filter, $S_{(-k_0-1,-k_1-1)}$ must be part of the same analysis filter. We will call ($S_{(k_0,k_1)}$, $S_{(-k_0-1,-k_1-1)}$) a conjugate pair. The support of each analysis filter consists of two conjugate pairs.

Permissibility. In the simplistic four-parallelogram filter banks, only the support of the lowpass analysis filter is determined. In the subband, due to decimation followed by expansion by $\mathbf{M}$, $H_0(\boldsymbol{\omega})$ has $J(\mathbf{M}) - 1$ images, which are shifted versions of $H_0(\boldsymbol{\omega})$ by

$$2\pi\mathbf{M}^{-T}\mathbf{k}, \mathbf{k} \in \mathbf{M}^{-T}$$

Adjacency of these images to the synthesis filter $F_0(\boldsymbol{\omega})$ result in edge and vertex aliasing. To make cancelation of these aliasing errors possible, the supports of the analysis filters in other subbands should be constrained appropriately. We would like to construct the other filters such that the configuration has edge and vertex permissibility at the same time. With this premise, we can show that supports of the other analysis filters $H_m(\omega)$ must contain

$$S_{(k_0,k_1)}, S_{(-k_0-1,-k_1-1)}, S_{(-k_0-1,k_1)}, S_{(k_0,-k_1-1)}. \quad (2)$$

This will in turn imply that the decimation matrix $\mathbf{M}$ is restricted to one of the following forms [7].

$$\text{case 1: } \mathbf{M} = \mathbf{U}\mathbf{\Lambda}_1, \text{ and case 2: } \mathbf{M} = \mathbf{Q}\mathbf{\Lambda}_2, \quad (3)$$

where $\mathbf{U}$ is unimodular, $\mathbf{Q}$ has $|\det \mathbf{Q}| = 2$ and $\mathbf{\Lambda}_1$ and $\mathbf{\Lambda}_2$ are diagonal. All matrices are integer matrices. Also with $\mathbf{N} = 2\mathbf{M}$, we can verify that no two terms in (2) are the same. Therefore each analysis filter contains four distinct shifts of the prototype.

Example 3.1. *Decimation matrix* $\mathbf{M}$ *not satisfying (3).* This example illustrates that some edge aliasing errors will remain uncancelled when (2) is not satisfied. Consider a four-parallelogram with decimation matrix

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}, \text{ and } \mathbf{N} = 2\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

In this case, $\mathbf{M}$ is not of the form in (3) and hence the analysis filters can not be of the form in (2). The supports of the analysis filters consist of four parallelograms, each a shifted version of $SPD(\mathbf{N}^{-T})$. The support of the lowpass analysis filter $H_0(\omega)$ is as shown in Fig. 9(a) and the support of $H_0(\omega)$ is $SPD(\mathbf{I})$ as shown in Fig. 9(b) (with frequency plane normalized by $2\pi\mathbf{N}^{-T}$). We have labeled all the squares in Fig. 9(b) by Q_k and Q'_k with (Q_k, Q'_k) denoting a conjugate pair. As $J(\mathbf{N}) = 12$, there are total 6 conjugate pairs (Q_k, Q'_k), for $k = 0, 1, \ldots, 5$. The lowpass filter $H_0(\omega)$ contains the pair (Q_0, Q'_0) and the pair(Q_1, Q'_1). The two images of Q_0 are at Q'_3 and Q'_4. The image at Q'_3 is edge adjacent to Q'_1. To cancel the aliasing from this image, it is necessary that the analysis filter contain both Q'_3 and Q_2. Let this analysis filter be $H_1(\omega)$, then $H_1(\omega)$ is constituted of the pairs (Q_2, Q'_2) and the pair (Q_3, Q'_3). In this case, the two pairs left for the last analysis filter $H_2(\omega)$ are (Q_4, Q'_4) and (Q_5, Q'_5). However, the image of Q_0 at Q'_4 is edge adjacent to Q_1 and results in edge based aliasing error. Cancelation of this aliasing requires that Q'_4 and Q'_2 belong to the same analysis filter, i.e. (Q_2, Q'_2) and (Q_4, Q'_4) belong to the same analysis filter. As the support of $H_2(\omega)$ consists of (Q_4, Q'_4) and (Q_5, Q'_5), this aliasing *can not be cancelled.*

4. Design of the Simplistic Four Parallelogram Filter Banks

Because $\mathbf{M}$ is of the special form in (3), the support described in (2) for four-parallelogram filter banks can always be obtained by cascading systems of low design cost. When $\mathbf{M}$ is as in case 1 of (3), the support configurations of the analysis and synthesis filters can be obtained by designing two 1D perfect reconstruction filter banks and performing a unimodular transformation as explained in [8]. For the second case of $\mathbf{M}$ in (3), the desired support configuration of four-parallelogram filter banks can be achieved by concatenating a separable 2D filter bank with a 2D two-channel filter bank in the form of a tree structure.

Case 1, $\mathbf{M} = \mathbf{U}\mathbf{\Lambda}_1$. The support configuration of a four parallelogram filter bank can be obtained by designing a $[\mathbf{\Lambda}_1]_{00}$-channel filter bank and a $[\mathbf{\Lambda}_1]_{11}$-channel filter bank. The reason is as follows. Observe that if a filter $P(\omega)$ has spectral support $SPD(\pi\frac{1}{2}\mathbf{U}^{-T}\mathbf{\Lambda}_1^{-1})$, then $P(\mathbf{U}^{-T}\omega)$ has spectral support $SPD(\pi\frac{1}{2}\mathbf{\Lambda}_1^{-1})$. The analysis filters $H_k(\omega)$ consist of four shifted copies of $SPD(\pi\frac{1}{2}\mathbf{U}^{-T}\mathbf{\Lambda}_1^{-1})$, so $H_k(\mathbf{U}^{-T}\omega)$ consists of four shifted copies of $SPD(\pi\frac{1}{2}\mathbf{\Lambda}_1^{-1})$. That is $H_k(\mathbf{U}^{-T}\omega)$ has the same support as that of an analysis filter in a separable filter bank. Design a separable filter bank of $J(\mathbf{M})$-channels by concatenating two 1D filter banks of $[\mathbf{\Lambda}_1]_{00}$ channels and $[\mathbf{\Lambda}_1]_{11}$ channels. Let $G_k(\omega)$ and $T_k(\omega)$ be the analysis and synthesis filters of this separable filter bank. Perform the following substitution,

$$H_k(\omega) = G_k(\mathbf{U}^T\omega), \quad F_k(\omega) = T_k(\mathbf{U}^T\omega), \quad (4)$$

then the analysis and synthesis filters of the 2D filter bank in Fig. 1 have the supports of four-parallelogram filter banks. This result is identical to those found earlier in [8].

Case 2, $\mathbf{M} = \mathbf{Q}\mathbf{\Lambda}_2$. Consider a tree structured filter bank. The first level of the tree is a two-channel filter bank with decimation matrix $\mathbf{Q}$. The lowpass analysis filter of the two-channel system has support $SPD(\pi\mathbf{Q}^{-T})$. For the second level of the tree, we use a separable $J(\mathbf{\Lambda}_2)$-channel filter bank that is obtained by cascading two 1D filter banks of $[\mathbf{\Lambda}_2]_{00}$ channels and $[\mathbf{\Lambda}_2]_{11}$ channels in the form of a tree. Then the overall filter bank has the desired supports of simplistic four-parallelogram filter banks.

Example 4.1. *Simplistic four-parallelogram filter banks.* Consider a 20 channel filter bank with decimation matrix

$$\mathbf{M} = \begin{bmatrix} 8 & -5 \\ -4 & 5 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}.$$

The matrix $\mathbf{M}$ has the form in case 1 of (2). Design a separable filter bank by concatenating a 1D four-channel filter bank and a 1D five-channel filter bank. Using the substitution in (4), the new nonseparable filter bank has the desired support as described in (2). The parallelogram prototype is as shown in Fig. 7. Fig. 10(a) shows the support configuration of the analysis filters. Fig. 10(b) shows the magnitude response of the lowpass analysis filter.

5. Other Possible Four Parallelogram Filter Banks

In the simplistic four-parallelogram filter banks, we have constrained $\mathbf{L}$ to be $2\mathbf{I}$ and the support of lowpass filter $H_0(\omega)$ to be a parallelogram of twice the size of the prototype parallelogram. However, for the most general four-parallelogram filter banks, the only requirement is that each filter contains four parallelograms of identical shapes. The two constraints in the simplistic four-parallelogram filter banks can be relaxed. The matrix $\mathbf{L}$ can be any integer matrix with $|\det \mathbf{L}| = 4$. Also the support of the lowpass analysis filter is not necessarily a parallelogram as in simplistic four-parallelogram filter banks. Consider the case that the support of $H_0(\omega)$ consists of four connected parallelograms. Namely, any one of the four parallelograms is edge adjacent or vertex adjacent to another parallelogram. It can be verified that, if $H_0(\omega)$ is aliasfree(M), the filter bank can not possess both edge and vertex permissibility, except in some special cases. One special case is the simplistic four-parallelogram filter banks in Sec. 3. In all the other special cases, the decimation matrix M is rather restricted and the determinant of M can not be arbitrary large. We will look at one such example.

Example 5.1. *A Permissible but not simplistic example.* Consider the filter bank in Fig. 1. Let

$$\mathbf{M} = \begin{bmatrix} 1 & 1 \\ -2 & 2 \end{bmatrix} \text{ and } \mathbf{L} = \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix}.$$

As $|\det \mathbf{M}| = 4$, the filter bank has 4 channels. In this case, the matrix $\mathbf{N} = 4\mathbf{I}$ and the parallelogram prototype is a sqaure. We choose the configuration as in Fig. 11. The four squares in the support of $H_k(\omega)$ are labeled as $S_{k,a}$, $S_{k,b}$, $S_{k,c}$, $S_{k,d}$, for $k = 0, 1, 2, 3$. When decimated and expanded by M, each of the four squares in the analysis filters has 3 images. These images result in edge and vertex aliasing. However, it can be shown that these edge and vertex aliasing errors appear in pairs and the configuration in Fig. 11 is edge and vertex permissible.

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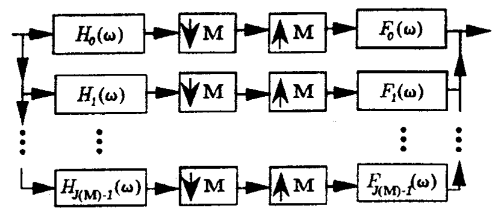


Fig. 1. $J(\mathbf{M})$-channel maximally decimated filter bank, where $J(\mathbf{M})=|\det(\mathbf{M})|$.

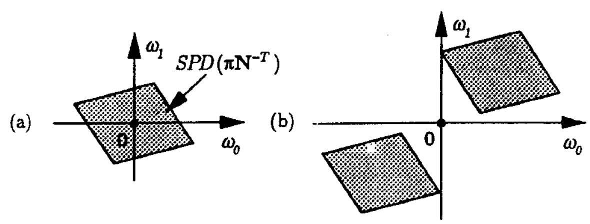


Fig. 2. Two-parallelogram filter bank. (a) Parallelogram prototype and (b) typical support of an analysis filter.

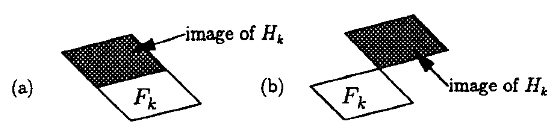


Fig. 3. (a) Image of the kth analysis filter is edge ajacent to the kth synthesis filter. (b) Image of the kth analysis filter is vertex ajacent to the kth synthesis filter.

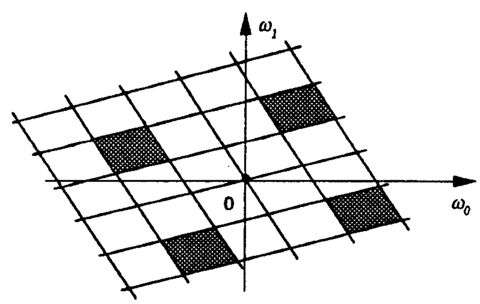


Fig. 4. Typical support of an analysis filter in a four-parallelogram filter bank.

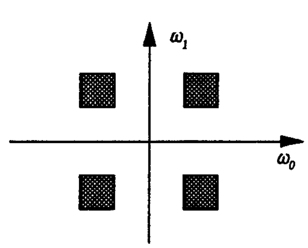


Fig. 5. Separable filter bank. Typical support of an analysis filter.

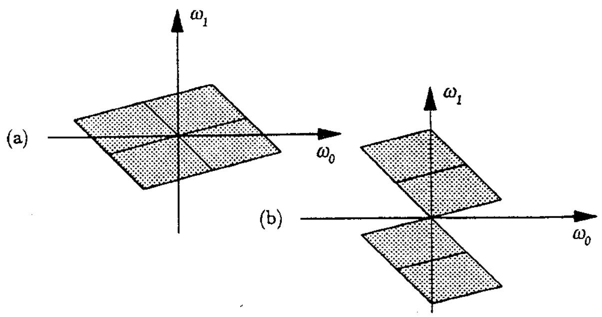


Fig. 6. Two of the possible supports for the 0th analysis filter.

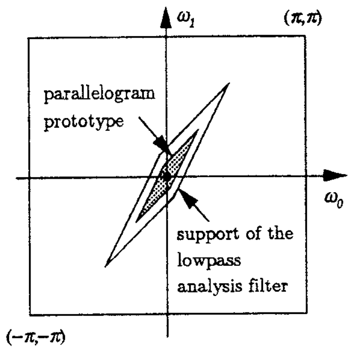


Fig. 7. Simplistic four parallelogram filter bank. The parallelogram prototype $SPD(\pi \mathbf{N}^{-T})$ and the support of the lowpass analysis filter $SPD(\pi \mathbf{M}^{-T})$.

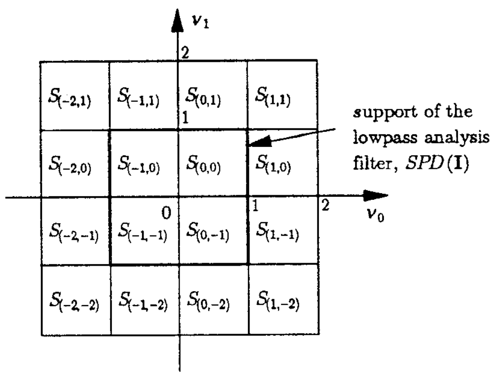
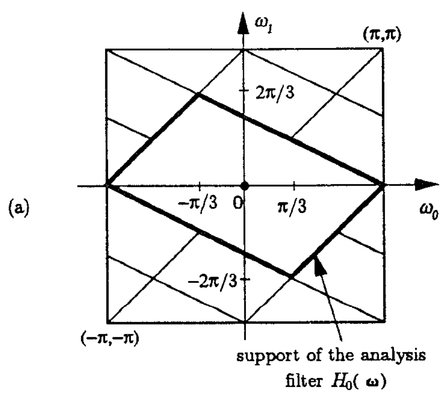


Fig. 8. The support of the lowpass analysis filter (with the frequency plane normalized by $2\pi \mathbf{N}^{-T}$).



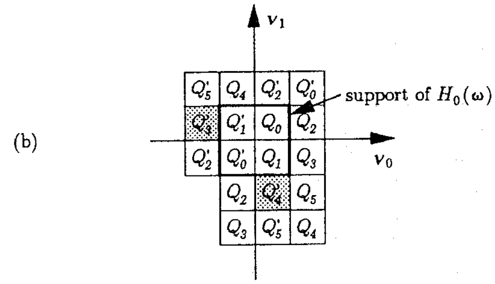


Fig. 9. Example 3.1. (a) Support of the analysis filters. (b) Pertaining to the illustration of alias cancelation (with the frequency plane normalized by $2\pi \mathbf{N}^{-T}$).

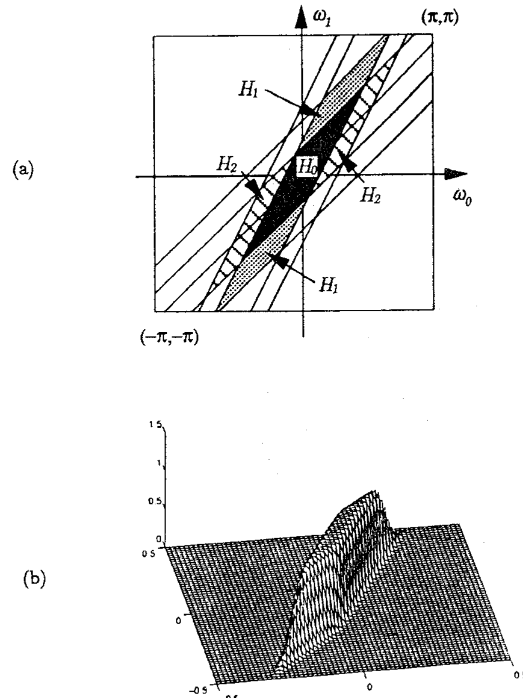


Fig. 10. Example 4.1. Simplistic four-parallelogram filter bank (a) Spectral support of the analysis filters and (b) the magnitude response of the lowpass analysis filter with frequency normalized by 2π.

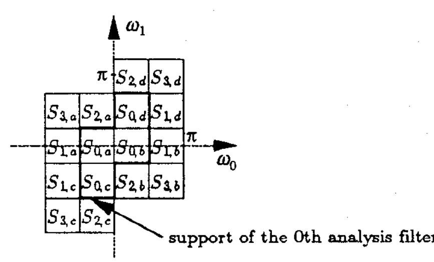


Fig. 11. Example 4.1. The support configuration of a permissible four-parallelogram filter bank that is not simplistic.