ANTIPODAL PARAUNITARY PRECODING FOR OFDM APPLICATION

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ABSTRACT

Paraunitary (PU) matrices have found many applications. In this paper, a special class of PU matrices, namely the antipodal PU (APU) matrices, is used as precoding matrices for OFDM systems. Both the zero-forcing and MMSE receivers will be derived for precoded OFDM systems with APU precoding matrices. The performance of such precoded OFDM systems will be analyzed. We will show that using a APU precoding matrix, we are able to average the noise variance in both the time and frequency domains, and this obtains time and frequency diversity. Experiments show that precoded OFDM systems with MMSE receivers have a much better bit error rate performance than the conventional OFDM system.

1. INTRODUCTION

Multirate systems and filter banks have played an important role in various areas of signal processing [1]. Of particular interest is the class of paraunitary (PU) matrices. One attractive feature of these matrices is their energy conservation property which can avoid the noise or error amplification problem. In the past, the design and complete parameterization of PU matrices have been successfully derived. In this paper, we are going to study a special class of PU matrices, namely the antipodal paraunitary (APU) matrices. An $M \times M$ polynomial matrix $\mathbf{T}(z) = \sum_{i=0}^{N-1} \mathbf{T}_i z^{-i}$ is APU if all the entries of $\mathbf{T}_i$ are ± 1 and it satisfies[1]

$$\widetilde{\mathbf{T}}(z)\mathbf{T}(z) = MN\,\mathbf{I}.$$

The tilde notation denotes $\widetilde{\mathbf{T}}(z) = \mathbf{T}^H(1/z^*)$, where ^{H} is transpose-conjugation and * is the complex conjugation. For the special case of constant (memoryless) matrices, APU matrices reduce to scaled Hadamard matrices. Various methods have been proposed for the construction of APU matrices [2] [3]. The application of APU matrices in synchronous spread spectrum communications has been explored [2] and promising results have been demonstrated.

In this paper, we will apply APU matrices to linearly precoded OFDM systems. Linearly precoded OFDM systems have been studied by a number of researchers [4] [5] [6]. When the OFDM system has a DFT precoding matrix, it was shown to be the same as the so-called the single carrier with frequency domain equalizer (SC-FDE) system, which was first introduced in [7]. In [4], it was shown that the SC-DFE system has the maximum diversity gain among all linearly precoded OFDM systems. In [5] [6], BER minimized precoder for OFDM system was considered. For high SNR transmission, the SC-FDE system is optimal in the sense that it minimizes the bit error rate among OFDM systems with any orthogonal precoding matrix. In these studies, the precoders are constant matrices and the resulting precoded OFDM systems belong to the class of block transmission systems.

OFDM systems with APU precoding matrices are overlapped block transmission systems; a block of data symbols is transmitted over several blocks of transmitted signals. By doing so, we are able to average the noise variance in both the time and frequency domains and this achieves time and frequency diversity. Both the zero-forcing and MMSE receivers will be derived. Performances of the proposed systems will be analyzed and compared with the conventional OFDM system.

2. PRECODED OFDM SYSTEMS

Fig. 1 shows the block diagram of a precoded OFDM system. In a precoded OFDM transmitter, the kth input block $\mathbf{s}(k)$ consisting of M modulation symbols, e.g. QAM symbols, are first passed through an M by M precoding matrix $\mathbf{T}(z)$. The output of $\mathbf{T}(z)$ is given by

$$\mathbf{u}(k) = \sum_{i=0}^{N-1} \mathbf{T}_i \mathbf{s}(k-i).$$

In this paper, we consider only APU precoding matrices. APU precoding matrices enjoy two main advantages. Firstly, they have very low complexity. Their implementation involves only additions and there exists an efficient butterfly structure for a broad class of APU matrices [3]. Secondly, as we will show later, APU matrices have the ability to average the noise variance over both the time and frequency domains. We assume that the APU precoding matrix $\mathbf{T}(z)$ is *normalized* so that $\widetilde{\mathbf{T}}(z)\mathbf{T}(z) = \mathbf{I}_M$. In other words, all the entries of $\mathbf{T}_i$ are $\pm 1/\sqrt{MN}$. After taking the M-point IDFT of $\mathbf{u}(k)$, we get:

$$\mathbf{x}(k) = \mathbf{W}^H \mathbf{u}(k),$$

where $\mathbf{W}$ is the $M \times M$ DFT matrix with its klth entry given by $[\mathbf{W}]_{kl} = 1/\sqrt{M} exp(-j2\pi kl/M)$. Before $\mathbf{x}(k)$ is transmitted, a cyclic prefix (CP) of length L is added. Note that unlike the conventional block transmission system, the transmitted block $\mathbf{x}(k)$ now contains information of N blocks of input vectors $\mathbf{s}(k-i)$

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[1] One can generalize the definition of APU matrices to include complex matrices. In this case, all the entries of the coefficient matrices $\mathbf{T}_i$ will have equal magnitude.

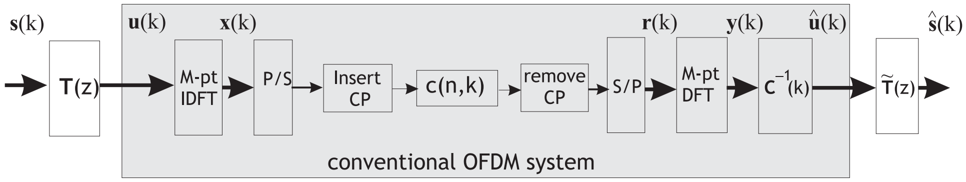

Fig. 1. An OFDM system with APU precoding matrix $\mathbf{T}(z)$.

for $0 \le i < N$. In this paper, we assume that the channel is slowly varying so that for each OFDM block, the channel response does not vary. We model the combined effect of DAC, transmit filter, channel, receive filter and ADC as an equivalent discrete time system with $c(n,k)$ denoting the nth tap of the impulse response when the kth block is sent. We also assume that the CP length L is large enough so that for all k, $c(n,k) = 0$ whenever $n > L+1$. The channel noise $\nu(n)$ is assumed to be an AWGN (complex) with variance N_0.

At the receiver end, the first L samples of the received block that correspond to the CP are discarded to remove the inter block interference. We obtain the $M \times 1$ vector $\mathbf{r}(k)$. Taking the DFT of $\mathbf{r}(k)$, we get

$$\mathbf{y}(k) = \mathbf{W}\mathbf{r} = \mathbf{C}(k)\mathbf{u}(k) + \boldsymbol{\nu}(k),$$

where $\mathbf{C}(k)$ is an $M \times M$ diagonal matrix whose (ℓ,ℓ)th entry is given by the DFT coefficient of $c(n,k)$:

$$C_\ell(k) = \sum_{n=0}^{L+1} c(n,k) e^{-j2\pi n\ell/M}. \quad (1)$$

The noise vector $\boldsymbol{\nu}(k)$ is an AWGN vector with autocorrelation matrix $N_0\mathbf{I}_M$. Assume that the channel does not have spectral null so that $\mathbf{C}(k)$ is invertible. After multiplying the diagonal matrix $\mathbf{C}^{-1}(k)$, we get

$$\widehat{\mathbf{u}}(k) = \mathbf{u}(k) + \mathbf{C}^{-1}(k)\boldsymbol{\nu}(k). \quad (2)$$

In the absence of channel noise, the vector $\hat{\mathbf{u}}(k) = \mathbf{u}(k)$ for all k. When the precoding matrix $\mathbf{T}(z)$ is PU, we can get a zero forcing receiver by taking $\widetilde{\mathbf{T}}(z)$ as the decoding matrix, as indicated in Fig. 1. Note that when we take $\mathbf{T}(z) = \widetilde{\mathbf{T}}(z) = \mathbf{I}_M$, the system in Fig. 1 reduces to the conventional OFDM system. It should be emphasized that even though the precoded OFDM system has an overlapping-block transmitter, the channel impulse response $c(n,k)$ can be different for different block number k and the system in Fig. 1 still has the zero-forcing property.

2.1. Noise Analysis

Define the noise vector in the kth block as

$$\boldsymbol{\beta}(k) = \widehat{\mathbf{u}}(k) - \mathbf{u}(k) = \mathbf{C}^{-1}(k)\boldsymbol{\nu}(k).$$

The autocorrelation matrices of $\boldsymbol{\beta}(k)$ are given by

$$\mathcal{R}_\beta(k,\ell) = \mathcal{E}[\boldsymbol{\beta}(k)\boldsymbol{\beta}^H(k-\ell)] = N_0\delta(\ell)\mathbf{C}^{-1}(k)\mathbf{C}^{-H}(k). \quad (3)$$

Because $\mathbf{C}^{-1}(k)$ is a diagonal matrix, we see from the above equation that $\boldsymbol{\beta}(k)$ is also an AWGN vector but each entry has a different variance.

Define the output noise vector $\mathbf{e}(k) = \widehat{\mathbf{s}}(k) - \mathbf{s}(k)$. Then it can be viewed as the output of $\widetilde{\mathbf{T}}(z)$ with the input vector $\boldsymbol{\beta}(k)$. Therefore, we can write

$$\mathbf{e}(k) = \sum_{\ell=0}^{N-1} \mathbf{T}_\ell^H \boldsymbol{\beta}(k+\ell).$$

Using the facts that $\boldsymbol{\beta}(k)$ is an AWGN vector and $\widetilde{\mathbf{T}}(z)$ is a normalized PU matrix, one can verify that its zeroth autocorrelation matrix at the kth block is given by

$$\mathcal{R}_e(k,0) = \mathcal{E}[\mathbf{e}(k)\mathbf{e}^H(k)] = \sum_{\ell=0}^{N-1} \mathbf{T}_\ell^H \mathcal{R}_\beta(k+\ell,0)\mathbf{T}_\ell,$$

where $\mathcal{R}_\beta(i,0)$ is the zeroth autocorrelation matrix of $\boldsymbol{\beta}(i)$ given in (3). Note that $\mathcal{R}_\beta(i,0)$ is a diagonal matrix. Looking at the ith diagonal term of $\mathcal{R}_e(k,0)$, we can write the noise variance at ith subchannel (when the kth block is being processed) as

$$\sigma_{i,\mathbf{T}}^2(k) = \frac{1}{N}\sum_{\ell=0}^{N-1}\Big[\frac{1}{M}\sum_{n=0}^{M-1}\frac{N_0}{|C_n(k+\ell)|^2}\Big], \quad (4)$$

where we have used (3) and the fact that all the entries of $\mathbf{T}_\ell$ have magnitude equal to $1/\sqrt{MN}$. The quantity $\sigma_{i,\mathbf{T}}^2(k)$ is independent of i; all subchannels have the same noise variance! Moreover the decoding matrix $\widetilde{\mathbf{T}}(z)$ has an averaging effect on the channel gains over a time period of N blocks. Note that we do not make any assumption about the APU matrix $\mathbf{T}(z)$. Any APU precoding matrix can achieve (4). From (4), we also see that the performance of the precoded OFDM with a zero-forcing receiver degrades significantly when some of the channel gains are small. The noise variances in all subchannels will be very large over a period of N blocks. To solve this problem, an MMSE receiver is needed and will be derived in the next section.

2.2. Comparisons with Other Systems

When we take $\mathbf{T}(z) = \mathbf{I}_M$, the system in Fig. 1 becomes the conventional OFDM system. In this case $\mathbf{u}(k) = \mathbf{s}(k)$. Thus, for the conventional OFDM system, we can obtain from (2) the output noise variance at the ith subchannel as

$$\sigma_{i,ofdm}^2(k) = \frac{N_0}{|C_i(k)|^2}. \quad (5)$$

The variance $\sigma_{i,ofdm}^2(k)$ depends on the block index k as well as the frequency index i, and it is inversely proportional to $|C_i(k)|^2$. For highly frequency selective channels, some of the gains $|C_i(k)|$ can be small and the performance of the OFDM system will be affected by these spectral nulls.

If we generalize the definition of APU matrices to include complex matrices, then the DFT matrix $\mathbf{W}$ is APU. When we take $\mathbf{T}(z) = \mathbf{W}$, i.e., the DFT matrix, the system in Fig. 1 becomes the SC-FDE system [7] [5]. By carrying out the same derivation, one can show that the noise variance of the SC-DFE system can be obtained by simply setting $N = 1$ in (4). The noise variance at the ith subchannel when the kth block is sent is given by

$$\sigma^2_{i,sc}(k) = \frac{1}{M}\sum_{n=0}^{M-1}\frac{N_0}{|C_n(k)|^2}. \tag{6}$$

Observe from the above expression that $\sigma^2_{i,sc}(k)$ is independent of the frequency index i. All the subchannels have the same noise variance and they are equal to the average noise variance of the conventional OFDM system.

We can clearly see the difference between the conventional OFDM, the SC-FDE and the precoded OFDM systems from the three expressions in (5), (6) and (4). Because the decoding matrix $\widetilde{\mathbf{T}}(z)$ is PU, it has the energy (or power) conservation property [1]. The average output noise variance for the three systems is the same. However they distribute these noise variances to the subchannels differently. For the conventional OFDM system, each subchannel can have a very different noise variance, especially when the channel is highly frequency selective. From (5), we see that subchannels having small $|C_n(k)|$ will suffer from large noise variances. On the other hand, the SC-FDE system has an average effect in frequency domain; it averages over all subchannels. For fast fading channel, $\sum_{n=0}^{M-1} 1/C_n(k)$ can vary from block to block and the performance of SC-DFE system will have a large variation with respect to k. From (6), we see that if $\sum_i |C_i(k)|^2$ is small for some k, the whole kth block will be severely affected by noise amplification problem. The precoded OFDM system with precoder $\mathbf{T}(z)$ has an averaging effect in both frequency and time-domain; it averages over all subchannels and over N OFDM blocks.

3. MMSE RECEIVER FOR PRECODED OFDM SYSTEMS

As we have mentioned earlier, in the presence of spectral nulls, precoded OFDM systems with zero-forcing receivers suffer from serious performance degradation. To avoid this problem, an MMSE receiver is needed. In the derivation of the MMSE receiver, we assume that the transmitted signals $\mathbf{s}(k)$ satisfy

$$\mathcal{E}\{\mathbf{s}(k)\mathbf{s}^H(k-\ell)\} = E_s\delta(\ell)\mathbf{I}_M.$$

In other words, the symbols are uncorrelated and have equal signal power. The fact that $\mathbf{T}(z)$ is normalized PU implies that $\mathbf{u}(k)$ also satisfies $\mathcal{E}\{\mathbf{u}(k)\mathbf{u}^H(k-\ell)\} = E_s\delta(\ell)\mathbf{I}_M$.

We assume that the receiver removes the first L samples corresponding to the CP so that there is no inter block interference. Given the received vector $\mathbf{r}(k)$, we want to design an MMSE receiver. As the DFT matrix $\mathbf{W}$ is invertible, there is no loss of generality if we consider the vector $\mathbf{y}(k) = \mathbf{W}\mathbf{r}(k)$. Given the vector $\mathbf{y}(k)$, we want to design an MMSE receiver. Consider an MMSE receiver (possibly time-varying) with N coefficient matrices $\mathbf{Q}(k,\ell)$ for $0 \le \ell \le N-1$. Given the input vector $\mathbf{y}(k)$, the output of the MMSE receiver can be described as:

$$\widehat{\mathbf{s}}(k) = \sum_{\ell=0}^{N-1}\mathbf{Q}(k,\ell)\mathbf{y}(k+\ell),$$

where $\mathbf{Q}(k,\ell)$ are $M \times M$ matrices. Our goal is to find $\mathbf{Q}(k,\ell)$ so that the following mean squared error is minimized.

$$\mathcal{E}\{\big(\widehat{\mathbf{s}}(k)-\mathbf{s}(k)\big)^H\big(\widehat{\mathbf{s}}(k)-\mathbf{s}(k)\big)\}.$$

Applying the orthogonality principle, one can verify that the MMSE solution is given by:

$$\mathbf{Q}(k,\ell) = \mathbf{T}_\ell^H\mathbf{\Lambda}(k+\ell),$$

where the diagonal matrix $\mathbf{\Lambda}(k)$ is given by

$$\mathbf{\Lambda}(k) = E_s\mathbf{C}^H(k)\Big(E_s\mathbf{C}(k)\mathbf{C}^H(k) + N_0\mathbf{I}_M\Big)^{-1}.$$

The nth diagonal entry of $\mathbf{\Lambda}(k)$ is given by

$$\lambda_n(k) = \frac{C_n^*(k)}{|C_n(k)|^2 + N_0/E_s}.$$

From the expression of $\mathbf{Q}(k,\ell)$, we see that the MMSE receiver can be decomposed into a time-varying diagonal matrix $\mathbf{\Lambda}(k)$ and the time-invariant matrix $\widetilde{\mathbf{T}}(z)$. Therefore, we can implement the MMSE receiver as Fig. 2. Comparing the zero-forcing and MMSE receivers in Fig. 1 and Fig. 2 respectively, one immediately sees that their only difference is the one-tap equalizer and they have the same implementational complexity. When there is no noise, i.e. $N_0 = 0$, the MMSE receiver reduces to the zero-forcing receiver.

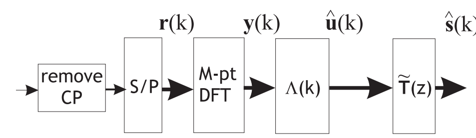

Fig. 2. An MMSE receiver for the precoded OFDM system.

One can verify that for the precoded OFDM system with an MMSE receiver, all the subchannels also have the same error variance and it is given by

$$\sigma^2_{mmse}(k) = \frac{1}{N}\sum_{\ell=0}^{N-1}\Big(\frac{1}{M}\sum_{n=0}^{M-1}\frac{N_0}{|C_n(k+\ell)|^2 + N_0/E_s}\Big).$$

From the above expression, it is clear that the decoding matrix $\widetilde{\mathbf{T}}(z)$ has an averaging effect in both frequency and time-domain; it averages over all subchannels and over N OFDM blocks. Moreover when some of the channel gains $|C_n(k)|$ approach zero, the error variance $\sigma^2_{mmse}(k)$ does not goes to infinity. In fact, the error variance is upper bounded by E_s. As we will see in the next section that by using a MMSE receiver, the performance of the precoded OFDM system is improved significantly.

4. SIMULATION

In this section, we carry out Monte-Carlo experiments to verify the performance of precoded OFDM systems with different precoders. The transmission channels are the modified Jakes fading channels described in [8]. In the experiments, we will use channel models with two different ratios of doppler frequency and transmission rate. A larger value of r indicates that the channel is changing faster. The ratio $r = 0.0001$ corresponds to a slowly varying channel whereas $r = 0.001$ corresponds to a channel that varies 10 times faster. The number of taps of the channels is 16. The channel noise $\nu(n)$ is AWGN with variance N_0. In our simulation, we assume that the receiver knows the exact channel response. The DFT size is $M = 64$ and the length of cyclic prefix is $L = 16$.

APU matrices of different length N will be used as the precoding matrices. When $N = 1$, the APU matrix reduces to the Hadamard matrix. It is known [5] that the OFDM system with a Hadamard precoding matrix has the same bit error rate performance as the SC-DFE system. The input vector $\mathbf{s}(n)$ consists of QPSK symbols with power equal to E_s. We plot the bit error rate curves versus SNR (signal to noise ratio), which is equal to E_s/N_0. In the simulation, we do not consider MMSE receiver for the conventional OFDM system because the bit error rate performance of OFDM systems with MMSE receivers is identical to that of OFDM systems with zero-forcing receivers.

The results for $r = 0.0001$ are shown in Fig. 3. From the figure, we see that the performance of precoded OFDM system with a zero-forcing receiver is worse than that of the OFDM system. This is because when the transmission encounters deep fading at some frequency bins, all the outputs of precoded OFDM receiver will be seriously affected by channel noise. On the other hand, for OFDM system, only a portion of the outputs will be seriously affected. However when an MMSE receiver is employed, the precoded OFDM systems have a much better performance than the OFDM system. If we compare the performance of precoded OFDM systems with different precoders, we see that when the channel is slowly varying, using a longer precoding matrix does not provide much gain in performance. This is because when the channel variation in the time domain is small, averaging the performance in the time domain has little effect on the performance.

For channel that is varying 10 times faster with $r = 0.001$, the results are shown in Fig. 4. Again we see that precoded OFDM system with a zero-forcing receiver does not perform well and using an MMSE receiver can greatly improve the performance of precoded OFDM systems. Also note that the performance improves as N (the length of the precoding matrix) increases. As the channel is fast varying, averaging in the time domain can provide additional gain. If we compare the cases of $N = 1$ and $N = 8$, averaging over 8 blocks can yield an additional gain of more than 2 dB when the bit error rate is 10^{-5}.

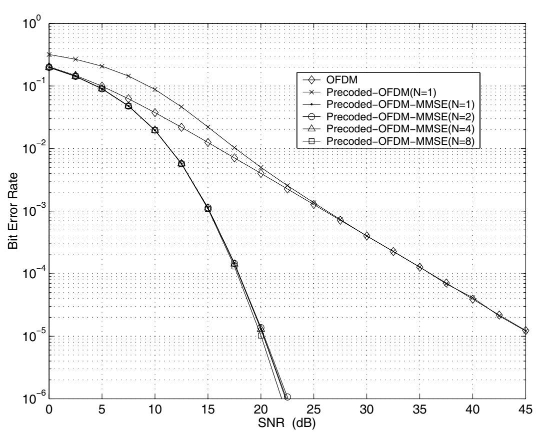

Fig. 3. Bit error rate performance for slowly varying channels

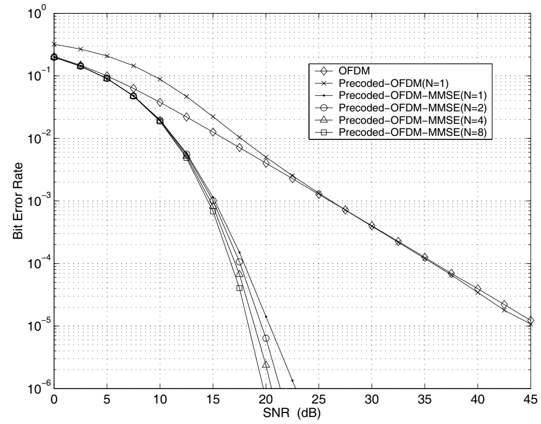

Fig. 4. Bit error rate performance for fast varying channels

5. CONCLUSIONS

In this paper, we have studied OFDM systems with APU precoding matrices. Using an APU precoding matrix, we can average the noise variances in both the time and frequency domains. We have derived MMSE receivers for precoded OFDM systems. Experiments show that precoded OFDM systems with MMSE receivers have a much better bit error rate performance that the conventional OFDM system.

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