NONSEPARABLE SAMPLING THEOREMS FOR TWO-DIMENSIONAL SIGNALS[†]

Yuan-Pei Lin and P. P. Vaidyanathan

Dept. Electrical Engr., Caltech, Pasadena, CA 91125

Abstract

It is well-known that continuous time bandlimited signals can be sampled without creating aliasing if the sampling period is small enough. It is also known that if $x(t)$ is a bandpass signal, the passbands of $X(\Omega)$ must be located properly for aliasfree maximal sampling. Similar situations arise in discrete time case. This paper addresses these issues for two-dimensional (2D) one- and two-parallelogram signals, which are respectively the classes of 2D signals (continuous or discrete time) whose Fourier transforms have supports consisting of one and two parallelograms. In this paper, we will derive necessary and sufficient conditions such that a one- or two-parallelogram signal (continuous and discrete time) allows maximal aliasfree sampling.

1. INTRODUCTION

If a one-dimensional (1D) signal $x(t)$ bandlimited to $[-\pi/N, \pi/N]$ (Fig. 1), $x(t)$ can be sampled without creating aliasing by a sampling period M, $M \leq N$; $x(t)$ is an aliasfree(M) signal. In this case $x(t)$ can be reconstructed from its samples . For a signal with total bandwidth $2\pi/N$, when the sampling period M is equal to N, we will refer to this as maximal sampling. If $x(t)$ is a bandpass signal with total bandwidth $2\pi/M$ as shown in Fig. 2, maximal aliasfree sampling of $x(t)$ depends on the relative positions of the two passbands as elaborated next.

Fact 1.1. *One-dimensional bandpass sampling theorem [1].* Suppose a 1D signal $x(t)$ has total bandwidth $2\pi/M$ as shown in Fig. 2 . Then $x(t)$ is aliasfree(M) if and only if ω_0 is a multiple of π/M, i.e., $\omega_0 = k\pi/M$, for some integer k.

More generally, if the support of $X(\Omega)$ does not overlap under modulo $2\pi/M$, $x(t)$ is aliasfree(M) [2]. Similarly, a D-dimensional signal $x(\mathbf{t})$ is aliasfree($\mathbf{M}$) for some $D \times D$ matrix $\mathbf{M}$ if the support of $X(\boldsymbol{\Omega})$ does not overlap under modulo $2\pi\mathbf{M}^{-T}$ [3].

[†] Work supported in parts by NSF grant MIP 92-15785, Tektronix, Inc., and Rockwell International.

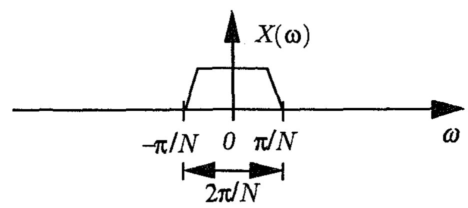

Fig. 1. Bandlimited signal with bandwidth $2\pi/N$.

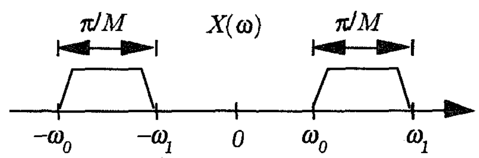

Fig. 2. Bandlimited real signal with total bandwidth $2\pi/M$.

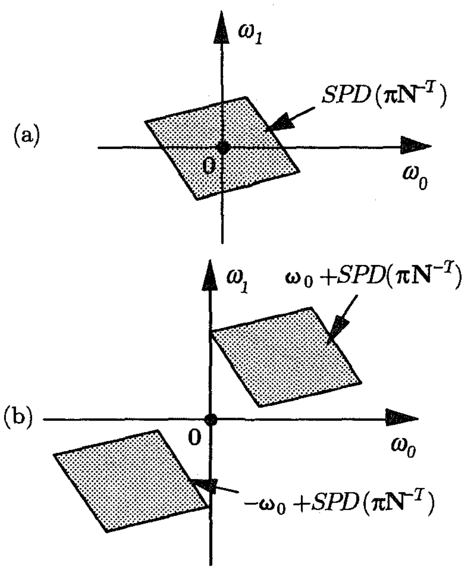

Fig. 3. Typical support of
(a) a one-parallelogram signal
(b) a two-parallelogram signal.

The issue of aliasfree maximal sampling is of great importance for many applications, e.g., filter bank design. In this paper, we focus on one subclass of two-dimensional (2D) signals, the one- and two-parallelogram signals (continuous and discrete time). We say a 2D continuous time signal $x(\mathbf{t})$ is a one-parallelogram (*One-P*) signal or two-parallelogram (*Two-P*) signal if its Fourier transform $X(\boldsymbol{\Omega})$ has a support consisting of one parallelogram (Fig. 3(a)) or two parallelograms (Fig. 3(b)). In Fig. 3 the symmetric parallelepiped $SPD(\mathbf{V})$ of a matrix $\mathbf{V}$ is the set

$$SPD(\mathbf{V}) = \{\mathbf{V}\mathbf{x}, \mathbf{x} \in [-1,1)^2\}.$$

Similarly, a discrete time signal $x(\mathbf{n})$ is called *One-P* or *Two-P* if the support of its Fourier transform $X(\boldsymbol{\omega})$ is the union of one or two identical parallelograms. For *One-P* or *Two-P* signals, there are no existing simple rules for testing aliasfree(M) property analogous to that given in Fact 1.1. The aim of this paper is to derive such rules (continuous and discrete time). The 2D bandpass sampling theorems for discrete time *Two-P* signals will particularly help us to get a clearer picture of 2D cosine modulated filter banks [4].

2. ONE-PARALLELOGRAM SIGNALS

Let $x(\mathbf{t})$ be a continuous time one-parallelogram (*One-P*) signal. It is well-known that if the support of $X(\boldsymbol{\Omega})$ is $SPD(\pi\mathbf{M}^{-T})$ for some matrix $\mathbf{M}$, $x(\mathbf{t})$ is aliasfree(M) [5]. Now consider the more general case that $X(\boldsymbol{\Omega})$ has support $SPD(\pi\mathbf{N}^{-T})$ or a shifted version of $SPD(\pi\mathbf{N}^{-T})$. The analysis of aliasfree(M) property of *One-P* signals is more intricate than that of 1D one-passband case. Let us define

$$\mathbf{L} \triangleq \mathbf{M}^{-1}\mathbf{N},$$

and denote the absolute value of the determinant of $\mathbf{L}$ by $|\mathbf{L}|$. The condition $|\mathbf{L}| \geq 1$ alone does not imply aliasfree(M) property [6] and a stronger condition is called for. In particular, the lattice of $\mathbf{L}^T$ (denoted by $LAT(\mathbf{L}^T)$) has to satisfy one additional property. More precise statement is given in the theorem to follow.

Theorem 2.1. *Continuous time sampling.* Let $x(\mathbf{t})$ be a continuous time *One-P* signal and the support of $X(\boldsymbol{\Omega})$ be $SPD(\pi\mathbf{N}^{-T})$ or a shifted version of $SPD(\pi\mathbf{N}^{-T})$, for some 2×2 matrix $\mathbf{N}$. Then $X(\boldsymbol{\Omega})$ is aliasfree(M) if and only if the matrix $\mathbf{L}$ defined as $\mathbf{L} = \mathbf{M}^{-1}\mathbf{N}$ satisfies

$$LAT(\mathbf{L}^T) \cap (-1,1)^2 = \{\mathbf{0}\}.$$

This necessary and sufficient condition means that no vector in $LAT(\mathbf{L}^T)$ is inside the square $(-1,1)^2$ except the vector $\mathbf{0}$. For example, let $\mathbf{L}^T = \begin{bmatrix} 1 & 0 \\ 0.5 & 2 \end{bmatrix}$, then $LAT(\mathbf{L}^T)$ is as shown in Fig. 4; $LAT(\mathbf{L}^T)$ has only one vector (the vector $\mathbf{0}$) inside the square $(-1,1)^2$. Notice that in 1D case, $|L| \geq 1$ if and only if $LAT(L^T) \cap (-1,1) = \{0\}$. However, this relation does not hold for the case of more than one dimension.

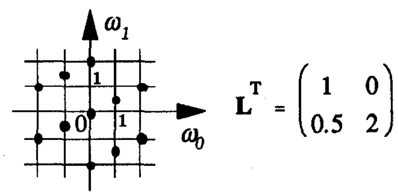

Fig. 4. Lattice of $\mathbf{L}^T$.

Proof of Theorem 2.1. Recall that $X(\boldsymbol{\Omega})$ is aliasfree(M) if and only if the support of $X(\boldsymbol{\Omega})$ does not overlap modulo $2\pi\mathbf{M}^{-T}$. So $X(\boldsymbol{\Omega})$ is aliasfree(M) if and only if, whenever $\mathbf{k} \neq \mathbf{0}$,

$$\boldsymbol{\Omega}_1 - 2\pi\mathbf{M}^{-T}\mathbf{k} \neq \boldsymbol{\Omega}_2, \quad \forall \boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \in SPD(\pi\mathbf{N}^{-T}).$$

Rearranging the above equation, we have $\boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_2 \neq 2\pi\mathbf{M}^{-T}\mathbf{k}$. As $\boldsymbol{\Omega}_1, \boldsymbol{\Omega}_2 \in SPD(\pi\mathbf{N}^{-T})$, $\boldsymbol{\Omega}_i$ can be expressed as $\boldsymbol{\Omega}_i = \pi\mathbf{N}^{-T}\mathbf{y}_i$, for some 2×1 vectors $\mathbf{y}_i \in [0,1)^2$, $i = 1, 2$. Hence

$$\boldsymbol{\Omega}_1 - \boldsymbol{\Omega}_2 = 2\pi\mathbf{N}^{-T}\mathbf{y}, \text{ for some } \mathbf{y} \in (-1,1)^2.$$

Using this expression, we have $\mathbf{y} \neq \mathbf{L}^T\mathbf{k}$, for $\mathbf{k} \neq \mathbf{0}$. This holds if and only if $LAT(\mathbf{L}^T) \cap (-1,1)^2 = \{\mathbf{0}\}$. ∎

This theorem can be generalized for signals of more than two dimensions, as the above proof can be carried out for any dimensions.

Discrete time case Suppose $x(\mathbf{n})$ is a discrete time *One-P* signal and the support of $X(\boldsymbol{\omega})$ is $SPD(\pi\mathbf{N}^{-T})$ or a shifted version of $SPD(\pi\mathbf{N}^{-T})$, for some 2×2 matrix $\mathbf{N}$. We can verify that $X(\boldsymbol{\omega})$ is aliasfree(M) for some integer matrix $\mathbf{M}$ if and only if $\mathbf{L} = \mathbf{M}^{-1}\mathbf{N}$ satisfies the condition $LAT(\mathbf{L}^T) \cap (-1,1)^2 = \{\mathbf{0}\}$.

3. CONTINUOUS TIME TWO-PARALLELOGRAM SIGNALS

Suppose $x(\mathbf{t})$ is a continuous time two-parallelogram (*Two-P*) signal. The support of $X(\boldsymbol{\Omega})$ consists of two parallelograms, each a shifted version of $SPD(\pi\mathbf{N}^{-T})$, as shown in Fig. 3(b). We are interested in the conditions such that maximal aliasfree decimation of $x(\mathbf{t})$ is allowed, i.e., there exists a 2×2 matrix $\mathbf{M}$ with $|\mathbf{M}| = |\mathbf{N}|/2$ such that $x(\mathbf{t})$ is aliasfree($\mathbf{M}$). The 1D bandpass sampling theorem hints that the two parallelograms in the support of $X(\boldsymbol{\Omega})$ should be somehow properly located. Indeed, whether $x(\mathbf{t})$ allows aliasfree maximal sampling is determined by the relative position of the two parallelograms.

Theorem 3.1. Let $x(\mathbf{t})$ be a continuous time *Two-P* signal and let the support of $X(\boldsymbol{\Omega})$ be the union of two parallelograms described by $\boldsymbol{\omega}_0+SPD(\pi\mathbf{N}^{-T})$ and $-\boldsymbol{\omega}_0 + SPD(\pi\mathbf{N}^{-T})$. Define $\mathbf{y}_0 = \mathbf{N}^T\boldsymbol{\omega}_0/\pi$, then $x(\mathbf{t})$ can be maximally decimated if and only if the following is true: *The vector* $\mathbf{y}_0$ *has at least one nonzero integer element.*

Proof of Theorem 3.1. *Necessity of the condition.* Recall that when we sample a signal $x(\mathbf{t})$ using the sampling matrix $\mathbf{M}$, the Fourier transform of the output is

$$\frac{1}{|\mathbf{M}|}\sum_{\mathbf{k}\in\mathcal{Z}^2}\widehat{X}_{\mathbf{k}}(\boldsymbol{\Omega}),\ \widehat{X}_{\mathbf{k}}(\boldsymbol{\Omega}) = X(\mathbf{M}^{-T}\boldsymbol{\Omega} - 2\pi\mathbf{M}^{-T}\mathbf{k}),$$

which consists of shifted and expanded versions of $X(\boldsymbol{\Omega})$. Observe that the two parallelograms of $\widehat{X}_{\mathbf{k}}(\boldsymbol{\Omega})$ are shifted versions of $SPD(\pi\mathbf{L}^{-T})$. In the case of aliasfree maximal sampling, the frequency plane will be filled by $\widehat{X}_{\mathbf{k}}(\boldsymbol{\Omega})$. So if $x(\mathbf{t})$ is aliasfree($\mathbf{M}$), the frequency plane is tiled by the parallelogram of $SPD(\pi\mathbf{L}^{-T})$.

For convenience, we normalize the frequency plane by $2\pi\mathbf{L}^{-T}$; the new axes ν_0 and ν_1 are the two entries of $\boldsymbol{\nu} = 2\pi\mathbf{L}^{-T}\boldsymbol{\Omega}$. After normalization the support of $\widehat{X}_{\mathbf{k}}(\boldsymbol{\Omega})$ appear as the union of two squares (Fig. 5), denoted by S and S' with $S = -\mathbf{y}_0/2+SPD(0.5\mathbf{I})$ and $S' = \mathbf{y}_0/2 + SPD(0.5\mathbf{I})$. So if the original frequency plane is tiled by the parallelogram of $SPD(\pi\mathbf{L}^{-T})$, the new normalized plane is tiled by the squares of $SPD(0.5\mathbf{I})$. In a square tiling, we can always observe at least one set of parallel lines (Fig. 6) and all the cells are bounded by these lines. For example in the tiling of Fig. 6(a), we can observe one set of parallel lines and all the squares are bounded by the horizontal lines. In the tiling of Fig. 6(b), however, we can observe vertical lines, and the squares are bounded by these vertical lines. So the passbands S and S' must be confined to these horizontal or vertical lines. As S and S' are separated by $\mathbf{y}_0$, $\mathbf{y}_0$ must have one integer element. One can verify that when $\mathbf{y}_0$ has one zero element, the other element of $\mathbf{y}_0$ must be an integer for square tiling to be possible.

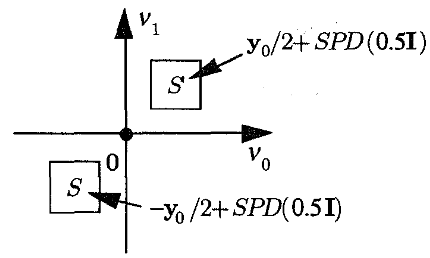

Fig. 5. Support of a two-parallelogram signal $X(\boldsymbol{\omega})$ with normalized axes.

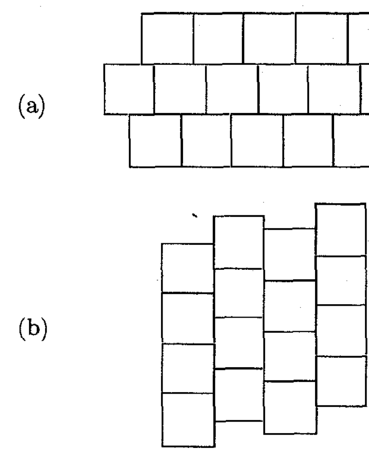

Fig. 6. Square tiling with (a) horizontal lines and (b) vertical lines.

Sufficiency of the condition. To show the condition is sufficient, we will construct a sampling matrix $\mathbf{M} = \mathbf{N}\mathbf{L}^{-1}$ with $|\mathbf{L}| = 2$ such that $x(\mathbf{t})$ is aliasfree($\mathbf{M}$). In particular, when $[\mathbf{y}_0]_0$ is a nonzero integer, we choose

$$\mathbf{L} = \begin{bmatrix} 1 & 0 \\ ([\mathbf{y}_0]_1 - 1)/[\mathbf{y}_0]_0 & 2 \end{bmatrix}.$$

When $[\mathbf{y}_0]_1$ is a nonzero integer, we choose

$$\mathbf{L} = \begin{bmatrix} 1 & ([\mathbf{y}_0]_0 - 1)/[\mathbf{y}_0]_1 \\ 0 & 2 \end{bmatrix}.$$

One can verify that with this choice of $\mathbf{M}$, $x(\mathbf{t})$ is aliasfree($\mathbf{M}$). ∎

Remark. The preceding theorem shows that maximal aliasfree decimation of a *Two-P* signal $x(\mathbf{t})$ completely hinges on the relative positions of the two parallelograms. However, for a given matrix $\mathbf{M}$ with $|\mathbf{M}| = |\mathbf{N}|/2$, whether $x(\mathbf{t})$ is aliasfree($\mathbf{M}$) depends not only on the relative positions of two parallelograms but also the matrix $\mathbf{L} = \mathbf{M}^{-1}\mathbf{N}$. In particular, it can be shown that $\mathbf{L}$ must have one integer row vector.

4. DISCRETE TIME TWO-PARALLELOGRAM SIGNALS

In this section, we discuss aliasfree decimation of discrete time *Two-P* signals. Like continuous time case, whether a *Two-P* signal $x(\mathbf{n})$ can be maximally decimated without creating aliasing depends on the location of the two parallelograms of $X(\boldsymbol{\omega})$. However, as the sampling matrix $\mathbf{M}$ is an integer matrix in discrete time case, additional conditions have to be imposed on the shape of the two parallelograms to ensure aliasfree maximal decimation. We first present the necessary and sufficient conditions (Theorem 4.1) such that, for a given sampling matrix $\mathbf{M}$, a discrete time *Two-P* signal $x(\mathbf{n})$ is aliasfree($\mathbf{M}$). From the theorem, we will observe the conditions that guarantee aliasfree maximal decimation of $x(\mathbf{n})$.

Theorem 4.1. Let $X(\boldsymbol{\omega})$ be a 2D *Two-P* signal and the support of $X(\boldsymbol{\omega})$ be the union of two parallelograms, each a shifted version of $SPD(\pi\mathbf{N}^{-T})$, where the matrix $\mathbf{N}$ is possibly a non integer matrix. Let $\mathbf{M}$ be an integer matrix with $|\mathbf{M}| = |\mathbf{N}|/2$. Then $X(\boldsymbol{\omega})$ is aliasfree($\mathbf{M}$) if and only if the following two conditions are satisfied [7].

1. Define $\mathbf{L} \triangleq \mathbf{M}^{-1}\mathbf{N}$, then $\mathbf{L}^T$ has the form $\mathbf{L}^T = \mathbf{\Gamma}\mathbf{U}$, where $\mathbf{U}$ is a unimodular matrix and $\mathbf{\Gamma}$ is of one of the following forms,
$$\underset{(a)}{\begin{bmatrix} 1 & \pm p \\ 0 & 2 \end{bmatrix}}, \quad \underset{(b)}{\begin{bmatrix} 2 & 0 \\ \pm p & 1 \end{bmatrix}}, \quad \underset{(c)}{\begin{bmatrix} 2 & \pm p \\ 0 & 1 \end{bmatrix}}, \quad \underset{(d)}{\begin{bmatrix} 1 & 0 \\ \pm p & 2 \end{bmatrix}},$$
where p is a real number and $0 < p \le 1$. This is equivalent to saying that $|\mathbf{L}| = 2$, $LAT(\mathbf{L}^T) \cap (-1,1)^2 = \{\mathbf{0}\}$ and $\mathbf{L}^T$ has one integer row vector.
2. Let $\boldsymbol{\omega}_0 = \pi\mathbf{N}^{-T}\mathbf{y}_0$, where $\mathbf{y}_0$ is a 2×1 vector. Corresponding to the above four cases of $\mathbf{L}$, $\mathbf{y}_0$ satisfies, (a) $[\mathbf{y}_0]_1$ is odd, (b) $[\mathbf{y}_0]_0$ is odd, (c) $\mathbf{y}_0 = \mathbf{L}^T\mathbf{k} + [1 \quad 0]^T$, (d) $\mathbf{y}_0 = \mathbf{L}^T\mathbf{k} + [0 \quad 1]^T$ for some integer vector $\mathbf{k}$.

For example, let $\mathbf{L}^T = \begin{bmatrix} 2 & 0 \\ 0.5 & 1 \end{bmatrix}$. One can verify that the first element of any vector $\mathbf{v} \in LAT(\mathbf{L}^T)$ is an integer and $\mathbf{L}$ satisfies the first condition. Notice the first condition is not necessary in 1D bandpass sampling theorem since $L = 2$ in 1D case and $LAT(L)$ consists of integers only. As indicated by the 1D bandpass sampling theorem in Fact. 1.1, in 1D case only the relative location of the two passbands needs to be constrained.

Remark. *Discrete time maximal decimation.* Theorem 4.1 provides the necessary and sufficient condition such that $x(\mathbf{n})$ is aliasfree($\mathbf{M}$) for a given integer matrix $\mathbf{M}$. A related question is, when does $x(\mathbf{n})$ allow aliasfree maximal sampling? From the above discussion, we know that for maximal decimation to be possible, $\mathbf{N}$ should be given by $\mathbf{N} = \mathbf{ML}$, for some $\mathbf{L}^T$ of the form given in Theorem 4.1. In other words, there must exist $\mathbf{L}$ with $|\mathbf{L}| = 2$ such that $\mathbf{M} = \mathbf{NL}^{-1}$ is an integer matrix.

References

[1] A. V. Oppenheim, A. S. Willsky, and I. Young, *Signals and systems*, Prentice Hall, 1983.

[2] V. Sathe and P. P. Vaidyanathan, "Effects of multirate systems on the statistical properties of random inputs," *IEEE Trans. on Signal Processing,* Jan. 1993.

[3] T. Chen and P. P. Vaidyanathan, "Recent developments in multidimensional multirate systems," *IEEE Trans. on CAS For Video Technology,* April 1993.

[4] Y. Lin and P. P. Vaidyanathan, "Two-dimensional paraunitary cosine modulated perfect reconstruction filter banks," Proc. ISCAS, April 1995.

[5] P. P. Vaidyanathan, *Multirate systems and filter banks*, Englewood Cliffs, Prentice Hall, 1993.

[6] E. Viscito and J. P. Allebach, "The analysis and design of multidimensional FIR perfect reconstruction filter banks for arbitrary sampling lattices," *IEEE Trans. on Circuits and Systems,* Jan. 1991.

[7] Y. Lin and P. P. Vaidyanathan, "Theory and design of two parallelogram filter banks," submitted to IEEE Trans. on SP, Oct. 1995.