TWO-DIMENSIONAL LINEAR PHASE COSINE MODULATED FILTER BANKS†

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Abstract

A filter bank is referred to as cosine modulated when all the analysis and synthesis filters are cosine modulated versions of one or two prototypes. The one-dimensional (1D) cosine modulated filter bank (CMFB) is well-known for low design cost and low complexity. In the application of subband coding, it is sometimes desirable that the individual filters have linear phase. In this paper, we will consider the class of 2D paraunitary CMFB with FIR linear phase filters. Necessary and sufficient conditions for perfect reconstruction of the 2D linear phase CMFB will be presented.

1. INTRODUCTION

In the context of one-dimensional (1D) filter bank design, the cosine modulated filter bank (CMFB) is well-known for low design cost and low complexity. The 1D CMFB has been studied extensively in the past (see [1] for references). In these systems, the analysis and synthesis filters do not have linear phase, which is a property sometimes desirable in image coding application. This motivates the development of a subclass of CMFB that have linear phase analysis and synthesis filters in [2].

Recently, there has been considerable interest in the design of two-dimensional (2D) maximally decimated filter banks (Fig. 1). In the coding of image or video data, it is of great importance that the filter bank can be implemented with low complexity and the features of CMFB are particularly attractive. The conventional 1D CMFB has been extended to 2D case, [3], [4]. In these 2D cosine modulated systems, the prototype filter has a parallelogram support, e.g., Fig. 2(a), and each analysis filter is a 2D cosine modulated version of the prototype. The support of each analysis filter consists of two identical parallelograms. Fig. 2(b) shows the supports of the analysis filters in a typical 2D CMFB system. It is shown in [5] that if the prototype filter has linear phase and the polyphase components of the prototype satisfy some 2D power complementary conditions, the 2D CMFB has perfect reconstruction. However, like its 1D counterpart, the 2D CMFB do not have linear phase analysis and synthesis filters.

In this paper, we will consider the class of 2D paraunitary CMFB with FIR linear phase filters. As the 2D CMFB given in [5], the support of the analysis and synthesis filters consist of two parallelograms. Necessary and sufficient conditions for perfect reconstruction will be presented. We will see that the paraunitary property of the 2D linear phase CMFB are completely determined by some pairwise power complementary relations of the polyphase components of the prototype filter. It turns out that these pairwise power complementary conditions are identical to those given in [5] and the prototype obtained therein can be used as the prototype for 2D linear phase CMFB.

2. REVIEW OF 1D LINEAR PHASE CMFB

Consider the filter bank in Fig. 1, in which the decimation ratio M is an even number. An M-channel linear phase CMFB is typically obtained by starting from a hypothetical M-channel uniform DFT filter bank, [1]. The DFT filters, denoted by $P_k(\omega)$, are shifted versions of a low-pass prototype filter $P_0(\omega)$ as shown in Fig. 3. The prototype $P_0(\omega)$ has bandwidth $2\pi/M$ and real coefficients. The filters $P_k(\omega)$ have complex coefficients except $P_0(\omega)$ and $P_{M/2}(\omega)$. Also the coefficients of $P_k(\omega)$ are the complex conjugate of those of $P_{M-k}(\omega)$, i.e.,

$$p_k(n) = p^*_{M-k}(n).$$

In the linear phase CMFB, the coefficients of the first $M/2+1$ analysis filters are the real part of $p_k(n)$, for $k = 0, 1, \cdots, M/2$, while the coefficients of the last $M/2-1$ analysis filters are the imaginary part of $p_k(n)$, for $k = 1, 2, \cdots, M/2-1$, except some scalar constants

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and delays. Precise expressions for the analysis filters are given below.

$$h_k(n) = 1/\sqrt{2}p_k(n), k = 0 \text{ and } M/2,$$

$$\begin{aligned} h_k(n) &= Re\{p_k(n)\}, \\ h_{k+\frac{M}{2}}(n) &= Im\{p_k(n - \frac{M}{2})\}, \end{aligned} \quad k = 1, 2, \ldots, \frac{M}{2} - 1.$$

So the analysis filters are the cosine (or sine) modulated versions of the prototype $P_0(\omega)$. The synthesis filters are time-reversed versions of the corresponding analysis filters except some delays,

$$f_k(n) = h_k(N + M/2 - n),$$

where N is the order of the prototype filter $P_0(\omega)$. With this setup, the analysis and synthesis filters have linear phase if the prototype $P_0(\omega)$ has linear phase.

Conditions for perfect reconstruction

Suppose the prototype filter $P_0(\omega)$ has linear phase and $p_0(n) = p_0(N-n)$, where N, the order of $P_0(\omega)$, is an odd multiple of $M/2$. Let $E_n(\omega)$ be the nth type 1 polyphase component of $P_0(\omega)$. Then the linear phase CMFB has perfect reconstruction if and only if $E_n(\omega)$ and $E_{n+M/2}(\omega)$ are power complementary, i.e.,

$$E_n(\omega)E_n^*(\omega) + E_{n+\frac{M}{2}}(\omega)E_{n+\frac{M}{2}}^*(\omega) = c,$$

$$n = 0, 1, \ldots \frac{M}{2} - 1, \text{ for some constant } c.$$

3. TWO-DIMENSIONAL LINEAR PHASE CMFB

Consider the filter bank in Fig. 1. The decimation matrix $\mathbf{M}$ is an integer matrix and $|\mathbf{M}|$ is an even number, where $|\mathbf{M}|$ denotes the absolute value of the determinant of $\mathbf{M}$. Similar to the construction of 1D linear phase CMFB, we start from a hypothetical $|\mathbf{M}|$-channel uniform DFT filter bank, [1]. The DFT filters $P_k(\omega)$ (Fig. 4) are shifted versions of a real-coefficient prototype $P_0(\boldsymbol{\omega})$,

$$P_k(\boldsymbol{\omega}) = P_0(\boldsymbol{\omega} - 2\pi\mathbf{M}^{-T}\mathbf{m}_k), \mathbf{m}_k \in \mathcal{N}(\mathbf{M}^T).$$

The prototype $P_0(\boldsymbol{\omega})$ has a parallelogram support described by $SPD(\pi\mathbf{M}^{-T})$, where the symmetric parallelepiped $SPD(\mathbf{V})$ of a matrix $\mathbf{V}$ is the set

$$SPD(\mathbf{V}) = \left\{\mathbf{V}\mathbf{x}, \mathbf{x} \in [-1, 1)^2\right\}.$$

So the prototype is nonseparable in general.

The analysis and synthesis filters

As in 1D linear phase CMFB, the analysis filters are the real and imaginary parts of the DFT filters. We first identify those pairs of DFT filters whose coefficients are complex conjugate of each other. It can be verified that

$$p_k(\mathbf{n}) = p_{k'}^*(\mathbf{n}), \text{ if } \mathbf{m}_{k'} + \mathbf{m}_k = \mathbf{0} \bmod \mathbf{M}^T.$$

Let us call $(\mathbf{m}_k, \mathbf{m}_{k'})$ a conjugate pair when $\mathbf{m}_{k'}$ and $\mathbf{m}_k$ are related in as above equation. In this case, if $k = k'$, $p_k(\mathbf{n})$ and $p_{k'}(\mathbf{n})$ are the same filter and hence $p_k(\mathbf{n})$ is real. So $p_k(\mathbf{n})$ has real coefficients whenever $2\mathbf{m}_{k'} = \mathbf{0} \bmod \mathbf{M}^T$. Let $\mathcal{N}_r(\mathbf{M}^T)$ denote the collection of such vectors. Recall in 1D, only two DFT filters have real coefficients. But in 2D case, there could be more than two real DFT filters depending on the matrix $\mathbf{M}$. For example, let $\mathbf{M} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$. Then the four DFT filters are respectively $p_0(n_0, n_1)$, $p_0(-n_0, -n_1)$, $p_0(n_0, -n_1)$ and $p_0(-n_0, -n_1)$; all four filters have real coefficients. More generally, one can verify that $\mathcal{N}_r(\mathbf{M}^T)$ contains either two or four vectors. Let $\mathcal{N}_i(\mathbf{M}^T)$ be the set that contains one vector from each conjugate pair $(\mathbf{m}_k, \mathbf{m}_{k'})$ and $\mathbf{m}_k \neq \mathbf{m}_{k'} \bmod \mathbf{M}^T$. It follows that $\mathcal{N}_r(\mathbf{M}^T)$ and $\mathcal{N}_i(\mathbf{M}^T)$ have no common vectors.

Now consider the following setup of analysis filters.

$$h_k(\mathbf{n}) = 1/\sqrt{2}p_k(\mathbf{n}), \mathbf{m}_k \in \mathcal{N}_r(\mathbf{M}^T),$$

$$\begin{aligned} h_k(\mathbf{n}) &= Re\{p_k(\mathbf{n})\}, \\ h_{k'}(\mathbf{n}) &= Im\{p_k(\mathbf{n} - \mathbf{M}[0.5\ 0.5]^T)\}, \end{aligned} \quad \mathbf{m}_k \in \mathcal{N}_i(\mathbf{M}^T),$$

where k and k' are such that $(\mathbf{m}_k, \mathbf{m}_{k'})$ is a conjugate pair. With such a setup the analysis filters are the 2D cosine (or sine) modulated versions of the prototype $P_0(\boldsymbol{\omega})$ and are nonseparable in general. The impulse responses of the synthesis filters are given by

$$f_k(\mathbf{n}) = h_k(-\mathbf{n}).$$

With the above construction of filters, we can verify that the analysis and synthesis filters have linear phase if the prototype has linear phase. Furthermore, the filter bank has perfect reconstruction when the prototype filter $P_0(\boldsymbol{\omega})$ is an ideal filter.

Theorem 3.1. *Necessary and sufficient condition for perfect reconstruction.* The 2D linear phase CMFB has perfect reconstruction if and only if the following three conditions are satisfied (see [6] for a proof).

(1) The lattice of $\mathbf{M}^T$ is a subset of the quincunx lattice.

(2) The prototype $p_0(\mathbf{n})$ has linear phase with $p_0(\mathbf{n}) = p_0(\mathbf{n}_s - \mathbf{n})$, where $\mathbf{n}_s = \mathbf{M}\,[0.5 \quad 0.5]^T$.

(3) Express the prototype in terms of polyphase representation,

$$P_0(\boldsymbol{\omega}) = \sum_{\mathbf{n}\in\mathcal{N}(\mathbf{M})} E_{\mathbf{n}}(\mathbf{M}^T\boldsymbol{\omega})e^{-j\boldsymbol{\omega}^T\mathbf{n}}.$$

Then the polyphase components $E_{\mathbf{n}}(\boldsymbol{\omega})$ satisfy the following power complementary conditions.

$$E_{\mathbf{n}}(\boldsymbol{\omega})E^*_{\mathbf{n}}(\boldsymbol{\omega}) + E_{\mathbf{n}'}(\boldsymbol{\omega})E^*_{\mathbf{n}'}(\boldsymbol{\omega}) = c,$$

where $\mathbf{n}$ and $\mathbf{n}'$ are related by $\mathbf{n} - \mathbf{n}' = \mathbf{n}_s \bmod \mathbf{M}$.

It turns out that these power complementary conditions are identical to those derived in [5]. So the prototype filters designed for the 2D CMFB in [5] can be used in the 2D linear phase CMFB as well. The design of the whole filter bank is reduced to designing a prototype filter with a parallelogram support.

4. DESIGN OF 2D LINEAR PHASE CMFB

It is well-known that the one-dimensional (1D) uniform DFT filter bank [1] can not have analysis filter and synthesis filters with good stopband attenuation except in ideal case. The reason for this is that with DFT type of frequency stacking, a considerable amount of aliasing will remain uncancelled if the individual filters have good stopband attenuation but not ideal, [7]. In this case, the support configuration for the analysis and synthesis filter is called nonpermissible. As support permissibility is a necessary condition for any successful design, we would like to examine the configuration of the 2D linear phase CMFB and check if a good design is possible.

As mentioned in Sec. 3, the 2D linear phase CMFB has prefect reconstruction when the prototype is an ideal filter. In practical cases, non-ideal roll-off of the filter causes aliasing. Particularly in the transition bands of the synthesis filters, different types of major aliasing is created. For example, in Fig. 5(a) one image of the analysis filter $H_k(\boldsymbol{\omega})$ is edge adjacent to the synthesis filter $F_k(\boldsymbol{\omega})$ and results in edge aliasing and similarly the image in Fig. 5(b) results in vertex aliasing.

When the configurations are not constructed properly, it is possible that some edge aliasing can not be canceled if the analysis and synthesis filters have good frequency selectivity. Such configurations are called edge nonpermissible, [8]. Similarly, if some vertex aliasing in a configuration are uncancelable when the filters have good frequency selectivity, the configuration is called vertex nonpermissible. For the individual filters to have good frequency selectivity, it is necessary that the 2D configuration have permissibility, which includes edge and vertex permissibility [8]. In this case, the importance of edge permissibility is much greater than vertex permissibility.

The configuration of the two-parallelogram CMFB constructed and designed in [5] has edge permissibility but lacks vertex permissibility. This imposes limitation on the stopband attenuation of the prototype filter. As the prototype of the 2D linear phase CMFB has to satisfy the same conditions as the prototype of the 2D CMFB in [5], the prototype of the 2D linear phase CMFB can not have good stopband attenuation and hence, the analysis and synthesis filters of the 2D linear phase CMFB can not have good frequency selectivity.

Example 4.1. *Two-dimensional linear phase CMFB.* Let

$$\mathbf{M} = \begin{bmatrix} 3 & -1 \\ 2 & 4 \end{bmatrix}.$$

The lattice of $\mathbf{M}^T$ is indeed a subset of the quincunx lattice. As $|\mathbf{M}| = 14$, the linear phase CMFB has 14 channels. Fig. 6 shows the magnitude response of the prototype filter whose polyphase components satisfy the power complementary conditions for perfect reconstruction. In this design example, each polyphase component of the prototype has four coefficients.

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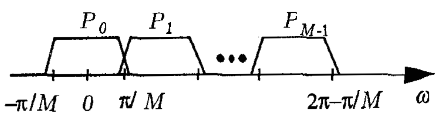

Fig. 3. An M-channel DFT filter bank.

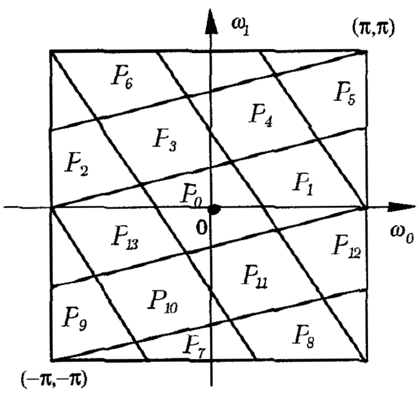

Fig. 4. Example of a 2D DFT filter bank.

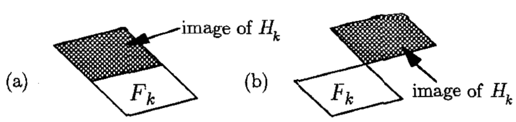

Fig. 5. (a) Image of the kth analysis filter is edge adjacent to the kth synthesis filter. (b) Image of the kth analysis filter is vertex ajacent to the kth synthesis filter.

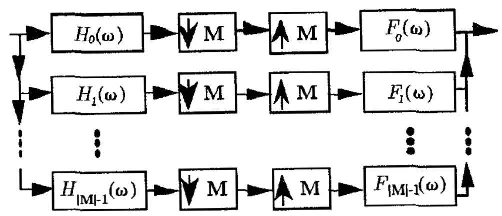

Fig. 1. $|\mathbf{M}|$-channel maximally decimated filter bank, where $|\mathbf{M}| = |\det \mathbf{M}|$.

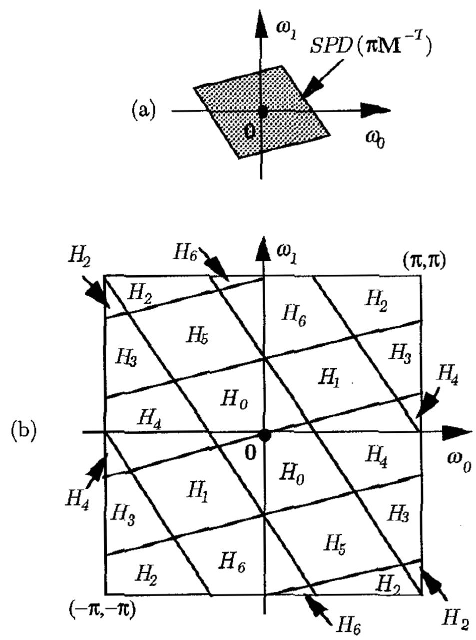

Fig. 2. A typical two-dimensional cosine modulated filter bank, (a) spectral support of the prototype, (b) support configuration of the filter bank.

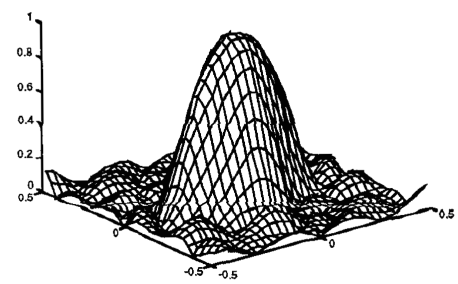

Fig. 6. Example 4.1. Two-dimensional linear phase cosine modulated filter bank. The magnitude response of the prototype with frequency normalized by 2π.