CONSIDERATIONS IN THE DESIGN OF OPTIMUM COMPACTION FILTERS FOR SUBBAND CODERS †

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Abstract

Recently there has been considerable interest in the design of optimal paraunitary filter banks for a given class of inputs. In this paper we address a number of practical considerations associated with the design and implementation of optimal paraunitary filter banks.

1. INTRODUCTION

In the design of subband coders, it is of interest to maximize the coding gain of the filter bank for a given class of input signals. For the case of paraunitary filter banks (PUFB), it is well known that the coding gain is the ratio of the arithmetic and geometric means of the subband variances [1]. It has recently been shown that this ratio is maximized if the analysis filters are such that the decimated subbands satisfy two properties, namely majorization and decorrelation [2]. It has further been shown that these two properties can be satisfied by designing each analysis filter to be an optimum energy compaction filter [3], [4], for an appropriate partial power spectrum defined from the input [5]. The study of optimal PUFB has therefore become an interesting topic recently.

In this paper we will discuss several new considerations in the design and implementation of these compaction filters. Of particular interest is the tree structure implementations of M-channel optimum PUFB, where M is a power of 2. We will show that the use of tree structures usually leads to a loss of coding gain. An example of optimal PUFB that is not realizable using tree structures will be given. We will present a coding gain formula for tree structures of PUFB, which will allow us to compute easily the coding gain increment for an additional split. We will also review a condition called permissibility, which is a property that has to be satisfied by the analysis filter passbands of any practical filter bank. We will show that optimum compaction filters do not, in general, satisfy this property. Thus the filter bank which maximizes the coding gain may not be of practical use in some cases.

We will make the same standard assumptions about input signals and subband quantizers as in [2]. For example, the input signal $x(n)$ will be assumed to be a zero-mean wide sense stationary process with power spectral density (psd) $S_{xx}(\omega)$. (See [2] for a more detailed description of these assumptions.)

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2. REVIEW OF OPTIMAL PARAUNITARY FILTER BANKS [2],[5]

Consider the M-channel filter bank in Fig. 1. Suppose the filter bank is paraunitary. With σ_x^2 denoting the input variance and $\sigma_{x_i}^2$ denoting the subband variances, the coding gain G is given by

$$G = \frac{\sigma_x^2}{\left(\prod_{i=0}^{M-1} \sigma_{x_i}^2\right)^{1/M}},$$

assuming optimal bit allocation. For a given input psd $S_{xx}(\omega)$, the variances $\sigma_{x_i}^2$ depend only on the analysis filters. If the filters are optimized such that the coding gain is maximized, the filter bank is called optimal.

Necessary and sufficient conditions for optimality. It has been shown that an M-channel PU filter bank is optimal for a given input if and only if the decimated subbands satisfy the following two properties.

1. The subband processes $x_i(n)$ are uncorrelated, that is, $E[x_i(n)x_k^*(m)] = 0$ for $i \neq k$, and for all n, m.
2. Suppose the subbands have been numbered such that $\sigma_{x_0}^2 \geq \sigma_{x_1}^2 \geq \cdots \geq \sigma_{x_{M-1}}^2$. Then for all ω,

$$S_{x_0x_0}(\omega) \geq S_{x_1x_1}(\omega) \geq \cdots \geq S_{x_{M-1}x_{M-1}}(\omega). \quad (1)$$

In this case, the set of power spectra $\{S_{x_kx_k}(\omega)\}$ is said to satisfy the majorization property.

For a fixed input psd, a filter bank satisfies these two properties has been successfully constructed. The construction process is greatly facilitated by the introduction of optimal compaction filters [3], which are discussed next.

Optimal compaction filters

Fig. 2 shows a filter $H(\omega)$ that can be viewed as an M-fold decimation filter. Consider designing $H(\omega)$ such that the output variance σ_y^2 is maximized subject to the constraint that $|H(\omega)|^2$ is Nyquist(M), i.e. $\sum_{k=0}^{M-1}|H(\omega-2\pi k/M)|^2 = M$, for all ω. The solution $H(\omega)$ will be called an optimum compaction filter. The construction of optimal solutions has been established in [3],[5]. The process is as follows.

For every frequency $\omega_0 \in [0, 2\pi/M)$ define the M alias frequencies $\omega_k = \omega_0 + 2\pi k/M$, where $0 \le k \le M-1$. Compare the values of $S_{xx}(\omega)$ at these M frequencies $\{\omega_0+2\pi k/M\}$. Let L be the smallest integer such that $S_{xx}(\omega_L)$ is a maximum in the set. Design

$$H(\omega_k) = \sqrt{M}\delta(k-L).$$

Then the filter $H(\omega)$, now completely defined on $[0, 2\pi)$, is the compaction filter for the input $S_{xx}(\omega)$.

Construction of optimal PUFB

It turns out the optimal PUFB for a given input can be obtained by solving successively M optimal energy compaction problems one at a time.

First we choose $H_0(\omega)$ to be the optimal compaction filter for $S_{xx}(\omega)$. Define a new psd $S_{xx}^{(1)}(\omega)$ by peeling off the part of $S_{xx}(\omega)$ that falls into the passband of $H_0(\omega)$, i.e.,

$$S_{xx}^{(1)}(\omega) = S_{xx}(\omega)(1 - H_0(\omega)/\sqrt{M}).$$

(The scale factor $\frac{1}{\sqrt{M}}$ is inserted for $H_(\omega) = \sqrt{M}$ in its passband.) Then choose $H_1(\omega)$ to be the optimal compaction filter for $S_{xx}^{(1)}(\omega)$. We continue this peeling off process $S_{xx}^{(k)}(\omega) = S_{xx}^{(k-1)}(\omega)(1-H_{k-1}(\omega)/\sqrt{M})$ and define the next analysis filter to the compaction filter for the partial spectrum $S_{xx}^{(k)}(\omega)$. It can be verified that the resulting filter bank is the optimal PUFB for the input $S_{xx}(\omega)$.

3. CODING GAIN OF TREE STRUCTURED PARAUNITARY FILTER BANKS

The coding gain of a tree structured filter bank (TSFB) can be expressed in terms of the coding gains of the member filter banks. For example, the coding gain G of the two-level TSFB (Fig. 3) is related to the coding gain, G_0, of the first level FB and the coding gains, G_1 and G_2, of the second level FB by $G = G_0\sqrt{G_{1,0}G_{1,1}}$.

Theorem 1. Consider the two-level TSFB in Fig. 3. Let the three member filter banks have coding gains respectively G_0, $G_{1,0}$ and $G_{1,1}$. Then the coding gain G of the overall TSFB is

$$G^{(dB)} = G_0^{(dB)} + \frac{1}{2}(G_{1,0}^{(dB)} + G_{1,1}^{(dB)}). \qquad (2)$$

Proof: By the coding gain formula for PUFB,

$$G = \sigma_x^2 \Big/ \left(\prod_{i=0}^{3}\sigma_{x_{1,i}}^2\right)^{1/4}$$

This can be rewritten as

$$G = \frac{\sigma_x^2}{\sigma_{x_0}\sigma_{x_1}}\left(\frac{\sigma_{x_0}^2}{\sigma_{x_{1,0}}\sigma_{x_{1,1}}}\right)^{1/2}\left(\frac{\sigma_{x_1}^2}{\sigma_{x_{1,2}}\sigma_{x_{1,3}}}\right)^{1/2}$$

We identify the three terms on the right hand side of this equation as G_0, $\sqrt{G_{1,0}}$ and $\sqrt{G_{1,1}}$, respectively. Writing the expression in dB, we arrive at (2). ∎

This result can be generalized to TSFB of more than two levels with member FB of more than two channels. For example, suppose $\mathcal{FB}_{1,1}$ in the second level has M channels and a further split is introduced to each subband. Let these M filter banks have coding gain $G_{2,0}, G_{2,1}, \ldots, G_{2,M-1}$. Then following a similar procedure we can show that the coding gain of the three-level TSFB is given by

$$G^{(dB)} = G_0^{(dB)} + \frac{1}{2}(G_{1,0}^{(dB)} + G_{1,1}^{(dB)}) + \frac{1}{2M}\sum_{i=0}^{M-1}G_{2,i}^{(dB)}.$$

The coding gain (dB) increment of the additional splits is $\frac{1}{2M}\sum_{i=0}^{M-1}G_{2,i}^{(dB)}$.

Remark. From the above expression, we can observe one property of the terminal FB (member FB that have no further split in their subbands). A terminal FB does not affect the coding gains of other FB in the previous levels. So to maximize the coding gain of the TSFB, it is necessary that the terminal FB be optimal for its input psd.

4. TREE STRUCTURE AND OPTIMAL PUFB

In this section we focus on the class of tree structured PUFB. First we present an example to show that the class of TSFB does not contain all the optimal PUFB. Using tree structure in general leads to a loss of coding gain.

An optimal PUFB that is not a tree

Consider an input psd as shown in Fig. 4(a). Fig. 4(b) shows the corresponding optimal analysis filters $H_0(\omega)$, $H_1(\omega)$, $H_2(\omega)$, and $H_3(\omega)$. Such a frequency stacking can not be achieved by using TSFB. To show this, suppose this is the overall analysis bank of a four-channel TSFB with two levels (Fig. 3). Because the first compaction filter $H_0(\omega)$ has support $[0, \pi/2)$, the analysis filter $H_{0,0}(\omega)$ of $\mathcal{FB}_0$ should contain $[0, \pi/2)$. For decimation of 2, the aliasing frequency of $\omega_0 \in [0, \pi/2)$ is $\omega_0 + \pi$, which falls into the range $[\pi, 3\pi/2)$. So $H_{0,0}(\omega)$ can not contain $[\pi, 3\pi/2)$; at most two of the four optimal filters can have energy in this region. But from Fig. 4(b) we see that three of the optimal filters have energy in this region. Therefore, the optimal filters can not be obtained from a tree structure.

The use of tree structure PUFB does not in general yield the maximum coding gain achievable by PUFB. Given a tree, suppose we design the member FB such that each is optimal for its input psd (i.e., tree structure of optimal building block PUFB). This, in general, will not yield the maximum coding gain for the given tree. Recall the construction of optimal compaction filters for a $M = 2^{m_0}$ channel PUFB. For each set of aliasing frequencies $\{\omega_k\}$, $M(M-1)/2$ comparisons among $\{S_{xx}(\omega_k)\}$ are required for majorization property in (1). In a tree structure of optimal PUFB, we can verify that only M comparisons are conducted among $\{S_{xx}(\omega_k)\}$, not enough for testing majorization condition.

Example 1. *Tree structure of optimal building block PUFB.* Suppose the input psd $S_{xx}(\omega)$ of the two-level TSFB Fig. 3 is as shown in Fig. 5(a). Consider the following two choices of TSFB.

(i) Let the FB in the first level be the optimal PUFB for the input $S_{xx}(\omega)$ and let the FB in the second level be respectively the optimal PUFB for $S_{x_0x_0}(\omega)$ and $S_{x_1x_1}(\omega)$. Fig. 5(b) shows the resulting TSFB analysis bank. The subbands variances are respectively 9, 4, 5, and 2. In this case the coding gain of the TSFB is $G^{(i)} = \frac{20}{(360)^{1/4}}$.

(ii) Choose the first-level analysis bank as in Fig. 5(c) and the second-level FB to be the optimal PUFB for $S_{x_0x_0}(\omega)$ and $S_{x_1x_1}(\omega)$. Then the subbands variances are respectively 9, 6, 3, and 2. The coding gain of the TSFB is $G^{(ii)} = \frac{20}{(324)^{1/4}}$, which is greater than $G^{(i)}$. In this case $G^{(ii)}$ is also the coding gain of the optimal PUFB.

Remark on optimal tree structured PUFB. In Example 1, suppose in the interval $(3\pi/4, \pi)$, the height of $S_{xx}(\omega)$ is 2 instead of 1. We can verify that tree structure of optimal PUFB is the optimal PUFB. Using the filters in Fig. 5(b) for the first level yields less gain. This shows that to obtain optimal tree structured PUFB, filters should be chosen not merely according to the values of $S_{xx}(\omega)$ at aliasing frequencies but according to the overall energy distribution.

Compaction filters for $M = M_1M_2$

The optimal compaction filter $H(\omega)$ (Fig. 2) for a composite integer $M = M_1M_2$ can be implemented by using the optimal compaction filter $H_1(\omega)$ for M_1 and the optimal compaction filter $H_2(\omega)$ for M_2 (Fig. 6). We first design the optimal compaction filter $H_1(\omega)$ for the input $S_{xx}(\omega)$ with respect to M_1. Then design the optimal compaction filter $H_2(\omega)$ for $S_{yy}(\omega)$ with respect to M_2. The product $H_1(\omega)H_2(M_1\omega)$ is the optimal compaction filter for the input $S_{xx}(\omega)$ with respect to M. The reason is as follows. The construction of optimal compaction filters in Sec. 2 indicates that we can think of compaction filters as a maximum selecting device. For every $\omega_0 \in [0, 2\pi/M]$, define the aliasing frequencies $\omega_{k,i} = \omega_0 + 2\pi k/M_1 + 2\pi i/M$. The optimal compaction filter $H(\omega)$ picks out a frequency ω_{k_0,i_0} such that $S_{xx}(\omega_{k_0,i_0})$ is a maximum of $\{S_{xx}(\omega_{k,i})\}$. In Fig. 6, the filter $H_1(\omega)$ first picks out a frequency $\omega_{k_0,i}$ such that $S_{xx}(\omega_{k_0,i})$ is a maximum of $\{S_{xx}(\omega_{k,i})\}$ for a fixed i. Then,

$$S_{yy}(M_1\omega_0 + 2\pi i/M) = S_{xx}(\omega_0 + 2k_0\pi/M_1 + 2i\pi/M).$$

Likewise, the filter $H_2(\omega)$ will single out a frequency ω_{k_0,i_0} such that $S_{xx}(\omega_{k_0,i_0})$ is a maximum of the set $\{S_{xx}(\omega_{k_0,i})\}$. It follows that $S_{xx}(\omega_{k_0,i_0})$ is a maximum of $\{S_{xx}(\omega_{k,i})\}$.

5. PERMISSIBILITY ISSUE

It is argued in [6] that, with certain frequency stacking in a filter bank, a considerable amount of aliasing will remain uncanceled if the individual filters have good attenuation. In this case, the support configuration is called nonpermissible.

The uniform DFT filter bank Fig. 5(b) is known to be a nonpermissible example whereas the cosine modulated type of stacking Fig. 7(a) is a permissible one. These two stackings are respectively the optimal four-channel PUFB for the following two cases: (i) The input is complex and the psd is monotone decreasing and (ii) the input is real and the psd is monotone decreasing. So optimal compaction filters in general are not permissible. However, the cosine modulated type of stacking Fig. 7(a) is not the only permissible stacking. For example in Fig. 7(a) consider swapping part of the supports of the first two filters (Fig. 7(b)). The supports of the other filter remain the same. The resulting stacking, clearly not cosine modulated type, is still permissible.

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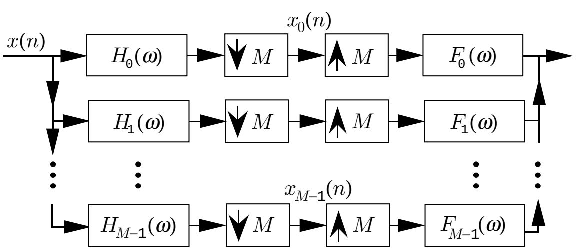

Fig. 1. An M-channel filter bank.

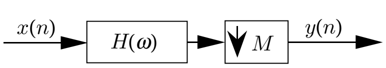

Fig. 2. Pertaining to the optimization of energy compaction problem

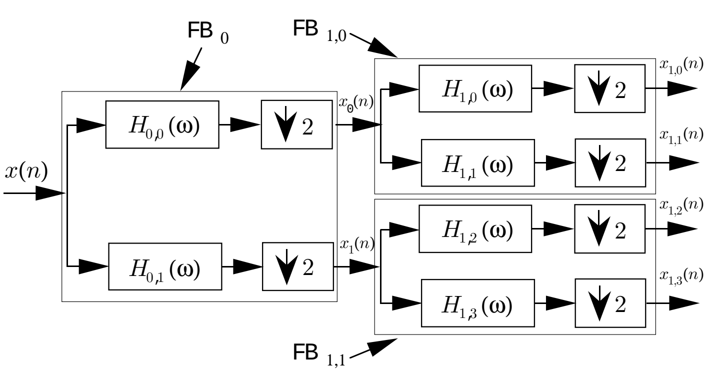

Fig. 3. Two-level tree structured filter bank.

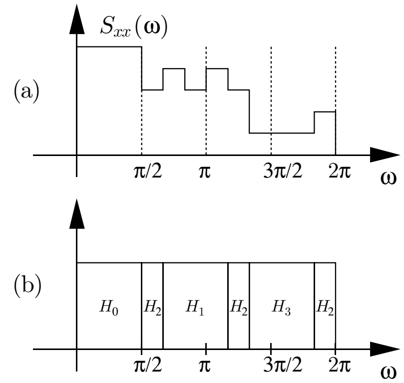

Fig. 4. An optimal paraunitary filter bank that can not be expressed as a tree. (a) Input power spectral density; (b) corresponding optimal paraunitary filter bank.

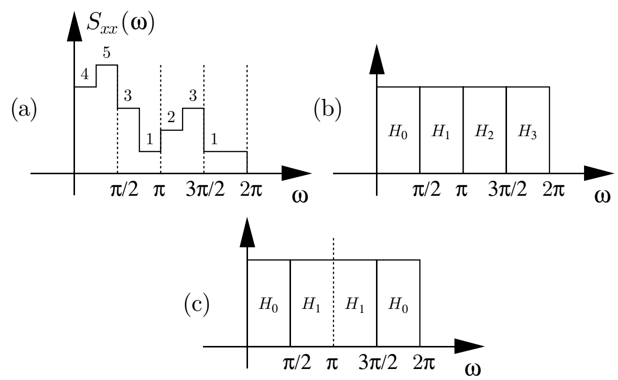

Fig. 5. Example 1. (a) Input power spectral density; (b) first set of overall analysis bank; (c) second set of analysis bank of first level.

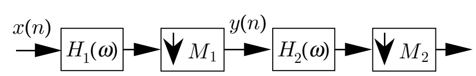

Fig. 6. Cascade implementation of optimal compaction filter for $M = M_1M_2$.

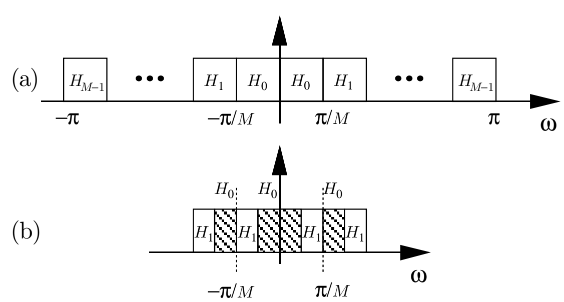

Fig. 7. (a) Cosine modulated type of stacking; (b) a permissible stacking.