AN ITERATIVE APPROACH TO THE DESIGN OF IFIR MATCHED FILTERS†

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Abstract

The interpolated FIR (IFIR) technique has been shown to be very useful in designing narrow band lowpass filters. The IFIR filters enjoys significant saving in the number of multipliers and can be implemented efficiently. In the case when an desired filter (non-IFIR) is given as in some applications, e.g., matched filters, we propose the use of IFIR filers to approximate the desired filter. An iterative approach to the design of IFIR approximation filters in the least square sense will be presented.

1. INTRODUCTION

An Interpolated FIR (IFIR) filter $P(z)$ is a filter of the form $G(z^L)F(z)$ [1]. The implementation of $P(z)$ can be done efficiently as the cascade of $G(z^L)$ and $F(z)$ (Fig. 1). The number of multipliers needed in the cascade implementation is approximately $1/L$ times that in direct implementation of $P(z)$. In addition, IFIR design also has advantages in the design phase. Traditionally the emphasis of the design has been given to considerations in the frequency domain, e.g., passband and stopband ripples; given the specifications, $G(z)$ and $F(z)$ can be separately design so that $P(z) = G(z^L)F(z)$ meets the specifications.

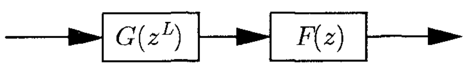

Fig. 1. Cascade implementation of IFIR filters.

For the application of discrete time matched filters in communications, a matched filter $h(n)$ is given [2]. The implementation advantage of IFIR filters can be exploited if $h(n)$ can be approximated by an IFIR matched filter $p(n)$. This is usually the case if the frequency response of the matched filter has a narrow band nature.

Let $h(n)$ be a given FIR filter with order N. Suppose we are to approximate it using an IFIR filter $p(n)$ of the form

$$P(z) = G(z^L)F(z),$$

for some appropriately chosen integer L to be discussed later. Then the coefficients of $p(n)$ are related to $g(n)$ and $f(n)$ by

$$p(n) = \sum_m g(m) * f(n - Lm).$$

Assuming $h(n)$ and $p(n)$ have the same order, then N_g and N_f, the order of $g(n)$ and $f(n)$ respectively, satisfy

$$LN_g + N_f = N.$$

The criterion of approximation to be considered here is the mean square error ϕ, given by

$$\phi = \frac{1}{N}\sum_n |h(n) - p(n)|^2.$$

The expression of ϕ consists of nonlinear terms of $g(n)$ and $f(n)$. To find optimum $f(n)$ and $g(n)$ that minimize the mean square error ϕ, complicated nonlinear optimization is inevitable.

On the other hand, if $f(n)$ is fixed, we can show that a unique $g(n)$ will minimize ϕ (to be shown in Sec. 2). Furthermore we will show that this solution can be obtained in closed form in terms of the coefficients of $h(n)$ and $f(n)$. Similarly for a fixed $g(n)$ we can find $f(n)$ in closed form such that ϕ is minimized. This motivates the idea of optimizing $g(n)$ and $f(n)$ iteratively. That is, we first initialize $f(n)$ and find the optimum $g(n)$, then we come back to optimize $f(n)$. This procedure can be repeated until a satisfactory ϕ is achieved.

2. THE ITERATIVE APPROACH

For convenience we first express all the functions in vector forms and relate them through matrix operations. Let

$$\mathbf{p} = \begin{bmatrix} p(0) \\ p(1) \\ \vdots \\ p(N) \end{bmatrix}, \mathbf{g} = \begin{bmatrix} g(0) \\ g(1) \\ \vdots \\ g(N_g) \end{bmatrix}, \mathbf{f} = \begin{bmatrix} f(0) \\ f(1) \\ \vdots \\ f(N_f) \end{bmatrix}$$

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Also let $g'(n)$ be the expanded version of $g(n)$, i.e.,

$$g'(n) = \begin{cases} g(n/L), & \text{if } n = Lk, \\ 0, & \text{otherwise.} \end{cases}$$

Then $p(n) = g'(n) * f(n)$ and $p(n)$ can be regarded as the output of the filter $f(n)$ when $g'(n)$ is the input. The relation $p(n) = \sum g'(m) * f(n-m)$ can be written in the matrix form $\mathbf{p} = \mathbf{F}\mathbf{g}'$, where $\mathbf{F}$ is of dimension $N \times (LN_g + 1)$ given by

$$\mathbf{F} = \begin{bmatrix} f(0) & 0 & \dots & 0 \\ f(1) & f(0) & \dots & 0 \\ \vdots & & \ddots & \\ f(N_f) & f(N_f - 1) & \dots & f(0) \\ 0 & f(N_f) & \dots & f(1) \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & f(N_f) \end{bmatrix}$$

The matrix $\mathbf{F}$ has LN_g+1 columns and each column consists of the vector $\mathbf{f}$ and LN_g+1 zero entries; we call $\mathbf{F}$ the expanded matrix of the vector $\mathbf{f}$ with respect to an input of length LN_g+1. Observe that

$$\mathbf{g}' = [g(0) \ \underbrace{0 \dots 0}_{L-1 \text{ zeros}} \ g(1)\ 0 \ \dots \ g(N_g)]^T$$

can be expressed as $\mathbf{g}' = \mathbf{S}\mathbf{g}$, where $\mathbf{S}$ is a $(LN_g + 1) \times (N_g + 1)$ matrix defined as

$$\mathbf{S} = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ 0 & 1 & 0 & \dots & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ 0 & 0 & 1 & \dots & 0 \\ & & \vdots & & \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix},$$

with $\mathbf{0}$ denoting the $(N_g - 1) \times 1$ zero vector. Then

$$\mathbf{p} = \mathbf{F}\mathbf{g}' = \underbrace{(\mathbf{FS})}_{\mathbf{A}}\mathbf{g}. \quad (1)$$

It follows that the mean square error ϕ is,

$$\phi = \frac{1}{N}(\mathbf{h} - \mathbf{A}\mathbf{g})^T(\mathbf{h} - \mathbf{A}\mathbf{g}). \quad (2)$$

It can be verified that both $\mathbf{F}$ and $\mathbf{S}$ have full rank and their product $\mathbf{A} = \mathbf{FS}$ also has full rank. This property ensures that an optimum $g(n)$ which minimizes the mean square error ϕ can be found in closed form, as we will see next.

Let us recall the following linear algebra problem [3]: Consider an $m \times 1$ vector $\mathbf{a}$ and a full rank matrix $\mathbf{T}$ of dimensions $m \times n$ and $m > n$. Then the quantity $(\mathbf{a} - \mathbf{Tb})^T(\mathbf{a} - \mathbf{Tb})$ is minimized if $\mathbf{b}$ is given by

$$\mathbf{b} = \mathbf{T}'\mathbf{a},$$

where $\mathbf{T}'$ is the left inverse of $\mathbf{T}$, defined as

$$\mathbf{T}' = (\mathbf{T}^T\mathbf{T})^{-1}\mathbf{T}^T.$$

By the same token, the mean square error ϕ as given in (2) is minimized if $\mathbf{g}$ is the vector $\mathbf{h}$ premultiplied by the left inverse of $\mathbf{A}$. Summarizing we have the following lemma.

Lemma 1. Let $h(n)$ be an FIR filter with order N. Suppose $p(n)$ is an IFIR filter of the same order and $P(z) = G(z^L)F(z)$, where $F(z)$ is a fixed filter. Then $\phi = \frac{1}{N}\sum |h(n) - p(n)|^2$ is minimized if

$$\mathbf{g} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{h},$$

where $\mathbf{A}$ is as defined in (1) ■

Even if $\mathbf{h}$ and $\mathbf{p}$ have large orders, $\mathbf{g}$ can have small order for large L; so the inversion of $\mathbf{A}^T\mathbf{A}$ is easy.

Similarly $p(n) = f(n) * g'(n)$ can be expressed as $p(n) = \sum g'(n-m)f(n)$. So the vector $\mathbf{p}$ also assumes the form

$$\mathbf{p} = \mathbf{B}\mathbf{f},$$

where $\mathbf{B}$ the $N \times (N_f + 1)$ expanded matrix of $g'(n)$ with respect to an input of length $N_f + 1$. It can be verified that $\mathbf{B}$ has full rank and the optimum solution of $f(n)$ is given by the closed form $\mathbf{f} = (\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T\mathbf{h}$. The design procedures can be summarized as follows.

Design procedures

1. Choose an appropriate L according to the magnitude response of $H(e^{j\omega})$.
2. Initialize $f(n)$.
3. Set $\mathbf{g} = \left(\mathbf{A}^T\mathbf{A}\right)^{-1}\mathbf{A}^T\mathbf{h}$.
4. Set $\mathbf{f} = \left(\mathbf{B}^T\mathbf{B}\right)^{-1}\mathbf{B}^T\mathbf{h}$.

After a few iterations, the changes in $g(n)$ and $f(n)$ are insignificant, then we stop iterating. Two comments on the initialization are in order.

1. *Choice of L.* Suppose $h(n)$ has stopband edge around π/M. Then L can be chosen as large as $3M/4$. In this case, the results of the optimization are very good in general and a mean square error smaller than 10^{-7} can be achieved in all our experiments.

2. The order of $f(n)$ has a significant effect on the results of optimization. A small N_f does not yield very good results; as a rule of thumb, an N_f in the range between $2L$ and $4L$ is usually appropriate. However, the initialization of $f(n)$ does not notably affect the convergence of optimization. A simple initialization, for example $f(n) = 1.0$, for $n = 0, 1, \ldots, N_f$, can be used.

Remark. The filters designed through this approach are IFIR. They approximate narrow band lowpass filters very well. But the savings yielded, however, depends on the bandwidth of the lowpass filters to be approximated.

3. EIGENFILTER DESIGN AND INCORPORATION OF FREQUENCY-DOMAIN CRITERION

The eigenfilter approach lends itself well to the incorporation of frequency domain considerations such as passband and stopband ripples [4][5]. We will formulate the least square problem in Sec. 2 as an eigenfilter problem and then derive an objective function ϕ' that reflects both time and frequency domain errors.

The first step of the problem is: For a given $\mathbf{f}$, how can we find $\mathbf{g}$ through eigenfilter approach such that $\mathbf{p}$ is a good approximation of $\mathbf{h}$? To facilitate the use of eigen technique, an additional condition will be incorporated in the optimization. More precisely, we will find $\mathbf{g}$ such that $\mathbf{p}$ approximates $\mathbf{h}$ in the least square sense subject to the constraint that $p(n_0) = h(n_0)$, where n_0 is such that $h(n_0) \neq 0$.

Let us define the error vector $\mathbf{e}$,

$$\mathbf{e} = \mathbf{h} - \mathbf{p}.$$

The mean square error is $\phi = \frac{1}{N}\mathbf{e}^T\mathbf{e}$. Because $p(n_0) = h(n_0) \neq 0$, the error vector can be written as $\mathbf{e} = p(n_0)\mathbf{h}/h(n_0) - \mathbf{p}$. Observe that $p(n_0) = \mathbf{v}^T\mathbf{p}$, where $\mathbf{v}^T$ is an appropriate vector. This relation gives the vector $\mathbf{e}$ the following form,

$$\mathbf{e} = \frac{1}{h(n_0)}\mathbf{h}\mathbf{v}^T\mathbf{p} - \mathbf{p} = \left(\frac{1}{h(n_0)}\mathbf{h}\mathbf{v}^T - \mathbf{I}\right)\mathbf{p}.$$

Using $\mathbf{p} = \mathbf{A}\mathbf{g}$ as in (1) we have

$$\mathbf{e} = \underbrace{\left(\frac{1}{h(n_0)}\mathbf{h}\mathbf{v}^T - \mathbf{I}\right)\mathbf{A}}_{\mathbf{Q}_g}\mathbf{g}. \qquad (3)$$

Therefore, the mean square error ϕ is

$$\phi = \frac{1}{N}\mathbf{g}^T\mathbf{Q}_g^T\mathbf{Q}_g\mathbf{g}. \qquad (4)$$

From the above quadratic form of $\mathbf{g}$, we see that ϕ is minimized if $\mathbf{g} = a\mathbf{t}$, where $\mathbf{t}$ is an eigen vector associated with the smallest eigen value of the matrix $\mathbf{Q}_g$. The value of a will be determined according to the constraint $p(n_0) = h(n_0)$.

In a similar manner, for a fixed $\mathbf{g}$, the mean square error ϕ can be expressed as the quadratic form of $\mathbf{f}$, say $\phi = \frac{1}{N}\mathbf{f}^T\mathbf{Q}_f^T\mathbf{Q}_f\mathbf{f}$. The optimization can be performed iteratively like in Sec. 2 until a sufficiently small ϕ is reached or the optimization converges. Notice that the solution of $g(n)$ thus obtained is different from the least square solution using the left inverse technique described in Sec. 2; the eigen solutions may not be as good as left inverse solutions.

Incorporation of frequency-domain criterion

For a given $\mathbf{f}$ the objectives in the passband and stopband can be expressed as quadratic forms of $\mathbf{g}$ similar to that in (4). The Fourier transform $E(e^{j\omega})$ of the error function $e(n) = h(n) - p(n)$ can be obtained as

$$E(e^{j\omega}) = \mathbf{e}^T\mathbf{c}(e^{j\omega}), \mathbf{c}(e^{j\omega}) = [\,1 \quad e^{-j\omega} \quad \ldots \quad e^{-Nj\omega}\,]^T$$

Suppose the filter $h(n)$ has passband edge ω_p and stopband edge ω_s. The amount of error in the passband is

$$\mathcal{E}_p = \int_0^{\omega_p} |E(e^{j\omega})|^2 \frac{d\omega}{2\pi} = \int_0^{\omega_p} \mathbf{e}^T\mathbf{c}(e^{j\omega})\mathbf{c}^\dagger(e^{j\omega})\mathbf{e}\frac{d\omega}{2\pi}$$

Using $\mathbf{e} = \mathbf{Q}_g\mathbf{g}$ in (3), we have

$$\mathcal{E}_p = \mathbf{g}^T\underbrace{\left(\int_0^{\omega_p} \mathbf{Q}_g^T\mathbf{c}(e^{j\omega})\mathbf{c}^\dagger(e^{j\omega})\mathbf{Q}_g\frac{d\omega}{2\pi}\right)}_{\mathbf{Q}_p}\mathbf{g}.$$

The amount of error in the stopband is

$$\mathcal{E}_s = \int_{\omega_s}^{\pi} |E(e^{j\omega})|\frac{d\omega}{2\pi}.$$

In a similar manner, expressing $\mathcal{E}_s$ as the quadratic form of $\mathbf{g}$, we have $\mathcal{E}_s = \mathbf{g}^T\mathbf{Q}_s\mathbf{g}$. Now we have an objective that incorporates the time and frequency domain error,

$$\phi' = \mathbf{g}^T(\alpha\mathbf{Q}_p + \beta\mathbf{Q}_s + (1 - \alpha - \beta)\mathbf{Q}_g^T\mathbf{Q}_g)\mathbf{g}.$$

It can be verified that all three matrices $\mathbf{Q}_p$, $\mathbf{Q}_s$ and $\mathbf{Q}_g^T\mathbf{Q}_g$ are symmetric and positive semi definite, so the solution of $g(n)$ that minimizes ϕ'

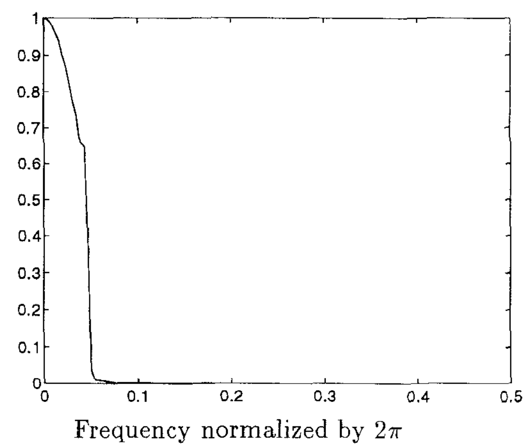
can always be identified. When $\mathbf{g}$ is given, similar expression for ϕ' can be derived and an optimum $\mathbf{f}$ can be obtained. As in previous section, iterative optimization can be continued until the solution converges.

4. DESIGN EXAMPLE

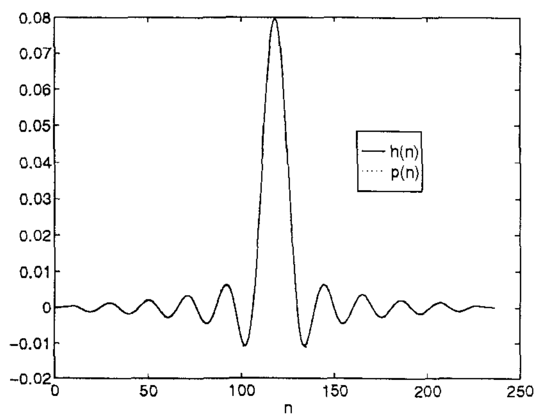
We now present an example to demonstrate that the proposed iterative left inverse optimization produce very good IFIR approximations of narrow band filters.

Example 1. *Iterative left inverse approach.* Consider a lowpass filter $h(n)$ of order $N = 236$ with magnitude response $|H(e^{j\omega})|$ as shown in Fig. 2(a). The stopband edge of $h(n)$ is around $\pi/10$. A choice of $L = 6$ or 7 should be appropriate as discussed in Sec. 2. We choose $L = 6$ and $N_f = 14$ ($N_f \approx 2.5L$), then we have $N_g = 37$. The impulse response of the resulting IFIR filter $p(n)$, after 25 iterations, is shown in Fig. 2(b) together with $h(n)$. The mean square error in this case is 2.7×10^{-8}. The magnitude responses of $P(e^{j\omega})$ and $H(e^{j\omega})$ are shown in Fig. 2(c); magnitude responses in dB scale are shown Fig. 2(d).

(a)



(b)



(c)

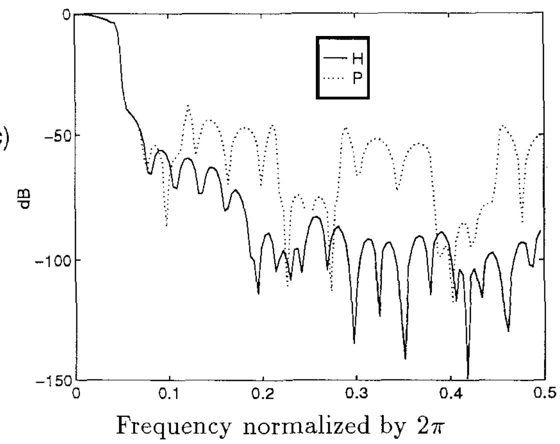


Fig. 2. (a) The magnitude response of the filter $H(e^{j\omega})$; (b) The impulse responses of $h(n)$ and the IFIR filter $p(n)$; (c) The magnitude response of $H(e^{j\omega})$ and $P(e^{j\omega})$ in dB.

The total number of multipliers needed is $N_f + N_g + 2 = 53$, which is around $\frac{1}{4}N$ in this case; the number of arithmetic operations needed is reduced approximately by a factor of 4. We see from the dB plots that, $P(e^{j\omega})$ has larger stopband ripples than $H(e^{j\omega})$. However in communication applications, like matched filtering, the time domain agreement (demonstrated in Fig. 2(b)) is far more important than frequency response quality. In the case when frequency response does need to be emphasized, a tradeoff between time domain error and frequency domain error can be provided by the eigen approach discussed in Sec. 3.

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